Resonance Effect, Critical and Resonance Velocities Applied to Diagnostics, Stability and Balancing Methods of Turbine and Generator Rotors over 40 MVA

Zlatan Racic, Marin Racic, **A.Y. Zhyvotov**, Y. G. Zhyvotov

Abstract

In power generation industry, diagnosing or balancing rotor vibration on large flexible rotors generally follows traditional applied rotor dynamics, where measured distributed rotor runouts are equated to "unbalances", which during rotor operation are assumed to excite a specific harmonic mode of the rotor at a specific operating angular velocity. In this paper it is clarified that the rotor behavior at the fundamental system critical velocity is not equivalent to harmonic resonance oscillation theory in physics. This system resonance behavior of a rotor in a rotor-bearing system encompasses very different behaviors and forces than what is seen within the external excitation of a "free" rotor's natural fundamental harmonic resonance at its inherent fundamental natural frequency. It is shown that what appears as a harmonic modal deflection is a rotating but non-oscillating static deformation that is created in an accelerated rotor under drive torque, formed by and driven by a rotating moment from the rotor's radially eccentric center of mass while gravity constrains the rotor in its journals in non-centroidal rotation. It is explained that in eccentric rotors, this system resonance phenomenon changes the location in space of the center of mass of the rotor body relative to the journal's rotation axis, as well as creates conditions for the self centering and reorientation of the rotor body mass axis once accelerated through its peak deflection at the system critical velocity, and this is often measured and observed as an unexpected drifting of the rotor shaft centerline position in its bearings. This process corresponds to a transfer of rotating and non-rotating reference frames through the critical velocity at approximately 90 degrees phase angle, at which point the rotor transitions to a "natural" centroidal rotation about its longitudinal center of mass axis. The observed angular response phase shift through the first system critical speed arises through conservation of angular momentum, and is not created just from system linear damping. This paper also determines the range of velocities of these rotor-bearing system "resonance" behaviors. The specific role of a rotor's inherent "free" resonance or internal eigenvectors and eigenvalues pertains primarily in design as a maximum stable system operating speed. The presented research is based on an inertial theory of rotor dynamics based on an equation of motion developed using impulse and momentum in an "open system" of continuous torque/energy input. The result of this research and understanding of rotor dynamics is of particular importance when developing balancing methods for rotors in the power generation industry for turbine-generator sets generating over ~40 MVA of electric power. Such an improved balancing method is also briefly presented, and it is recommended that such an approach be implemented in industry and incorporated within current balancing standards as a separate subsection for these very large flexible rotors. The equations and plots in this paper were originally presented by A.Y. Zhyvotov at the 13th World Congress in Mechanism and Machine Science in Guanajuato, Mexico in 2011, and this paper heavily expands upon that theoretical presentation.

Keywords: Critical Velocity, Natural Resonance, Stability, Reference Frames, Constraints, Balancing

1. Introduction

The standard for diagnostics and balancing specifically of large flexible turbine and generator rotors in power generation (over ~40 MVA) with "significant" eccentricities or a bow (with an eccentric mass axis of > ~2mils) should receive their own categorical approach and treatment as a special subgroup for solving vibrations while in service shops. This should include modified, improved balancing methods on balancing machines, and should be applied in resolving vibrations and stability issues in operation, separately from the well established “industry standard” practical approaches on all other high speed rotating machines.

For new rotors within accepted tolerances for residual eccentricities, existing standards are well proven and successful, even for these large rotors. However, “service industry established standards” are insufficient and less than optimized for the specific cases of large, flexible turbine/generator rotors with significant distributed mass eccentricity. The development of a new branch or subsection of standards applies especially to this unique segment of rotors which in practice creates the great majority of disruption in operation and cost to users in the power generation industry.

Note that in order to explain and understand *why* this special treatment and modified approach is necessary and why it is successful, it requires a fundamental and novel review of rotor dynamic processes that occur (or that are uniquely applicable) in these large, flexible and eccentric/bowed rotors in the service industry. The conclusions in this paper are derived from both experiment and analytical verification, considering a rotor as a continuous solid body (not a "point mass" superposition) with eccentric mass axis, constrained by gravity on flexible supports, as a horizontal rotating machine viewed as a rotor-bearing system. From the results of the study, and in comparison with commonly applied industry standard processes when dealing with such turbine and generator rotors undergoing service work and balancing, it is necessary to present a number of key differences in the assumptions and understanding of what is physically occurring in the rotor, and differences in the analysis and diagnostics of real data collected from vibration sensors, and in its practical interpretation and application.

In rotating machines, rotor vibration is commonly understood and visualized as a flexible linearly oscillating deformation of a beam (per a mass-spring-damper model), similar to a statically tensioned excited harp string exhibiting a modal shape, except that in a rotating machine as a system, it happens to also be rotating by an external torque force. This motion is also typically considered only in the rotor alone. However, to best understand rotating machine vibration and to best apply the understanding in design and in diagnostics, troubleshooting and balancing, rotors (and their measured “vibration”) should be looked at only within the context of a rotor as a continuous solid body interacting with supports within a combined rotor/bearing system. Here we must make a distinction between a rotor‘s fundamental natural harmonic oscillation (longitudinal velocity wave such as when it is hanging “free” and impacted and excited by a impulse force, orthogonal to rotor mass axis, and sufficiently large proportional to rotor mass), versus pseudo-static rotor deflection. In pseudo-static deflection, the rotor may bend/deflect from internal asymmetric “centrifugal” forces (reactive to centripetal forces) and will synchronously rotate (precess) in this bent/deflected shape as a “rotating banana” without oscillating linearly as a plucked guitar string. This “static” deflected shape is analogous to the fundamental harmonic mode shape, but without harmonic oscillation, even though vibration proximity sensors read the motion of the measured rotor surface as sine wave (which as a mathematical creation from signals from two orthogonal transducers presents the proper orbital motion).

The rotating machine must be viewed as a system, and the system is best represented as acting as an energy converter in an open system, converting continuous torque (power) input into circular motion of the rotor mass. Understanding the vibrations requires recognizing its root source as a form of this energy conversion, within an accelerated rotor mass from a state of rest to an angular velocity determined as the rotor system "critical velocity". An open system model is important because the presented rotor behaviors depend on the presence of continuous torque input and rotor acceleration. When an eccentric rotor is accelerated, the output of the energy conversion process is a combination of the desired kinetic energy converted to useful work, and the undesired lost energy which is manifested in rotor "vibration". This lost energy appears as kinetic energy as a translational orbital motion in "space" (and also absorbed via an increased moment of inertia to maintain rotation speed), and appears as potential energy in the form of internal rotor strain and as temperature and cyclic pressure transmitted to the bearings and supports.

It is helpful to view a continuous rotor body as a sum of internal molecules with binding energy based on the rotor's material atomic structure**.** If an accelerating rotor is ideally "rigid", then the internal mass binding energy does not allow any external deformation (bending between constraints), and the total added energy is converted to synchronous oscillating pressure on the rotor supports. If an accelerating rotor is "flexible", (i.e. deflects/sags > ~2 mils under gravity at rest, as most large rotors in practice do), then as the rotor is accelerated from a state of rest, it will still behave as "rigid" up to the rotor angular velocity corresponding to the beginning of rotor system critical velocity range, but from this point on the rotor will undergo a pseudo-static deformation. The sum of axially distributed centrifugal forces in purely "rigid" rotors is 100% transmitted to the journal constraints. The sum of centrifugal forces of axially distributed centrifugal forces in "flexible" rotors is proportioned between forces transmitted to the journal constraints and into rotor bending/deformation up to the system critical velocity at ~ 90 degrees phase, based on the rigidity of the rotor material.

Within this energy conversion model is the phenomenon through the system critical velocity range, of a process where a rotor with any amount of body mass eccentricity transitions from a forced non-centroidal rotation to a "natural" centroidal rotation about its inherent longitudinal center of mass axis. This transition or "critical speed" process is the reason why a rotor in an appropriately designed rotor-bearing system can operate safely at speeds above its fundamental system critical velocity. In the very first days of rotating machine design in the 19th century, based on a simple model of circular motion of a mass on a spring, it was hypothesized that no machine could operate above its critical speed, predicting that the centrifugal forces of "unbalance" would continually grow with increasing rotation speed and destroy the machine. Of course this concern was proven false, but the full explanation why, has not been shown in fullest detail, with much of the explanation historically attributed to the affects of damping in the system, founded on a modeled representation of linear oscillation. However, while damping provides an excellent and predictive mathematically modeled explanation, it has not been shown that external damping alone truly represents the complete physics of the system critical velocity resonance phenomenon in horizontal rotating machines.

2. Key Points in the Behavior of Large, Flexible Eccentric Rotors

Presented here is first a qualitative discussion of the behavior of large flexible rotors with significant distributed mass eccentricity, and the application of these behaviors toward rotor balancing for these cases. Following this section is a theoretical and mathematical presentation of the rotor behavior and determination of the specific parameters governing it. Note that in this paper in forthcoming descriptions, the term "precession" is used solely to indicate the circular translation in space of the rotor mass axis about the rotor journals (or spin) axis within the solid rotor body, and is not referring to precession as the circular "wobble" of the orientation of an axis of rotation. Similarly, the term "whirling" is used to distinguish and indicate the circular translation of the rotor journal (or spin) axis about the rotor mass axis.

Rotor behavior with mass eccentricity

2.1. ***Translation orbit vs. linear harmonic oscillation*.** Rotating machine vibration at the first system critical speed is not a cyclic modal harmonic linear oscillation of the rotor excited by rotor "unbalance". Instead it is a synchronous precession of the quasi-static (non-oscillating) rotor body as a result of axially distributed masses along the rotor body between its bearings. These masses form a centroidal axis radially offset or skewed relative to the journal centerline axis, which during rotor acceleration leads to an orbit amplitude rise due to the prevention of the rotor achieving a "natural" centroidal rotation. This longitudinal centroidal axis (mass axis) is precessing synchronously around the journal axis when a rotor is accelerated from a state of rest until reaching its first system critical velocity at ~ 90 degrees phase. The resulting lateral translation of the journals (as orbital motion), and increasing radius of orbital motion during rotor acceleration, comes from gravity holding and forcing the rotor to spin about its journal centerline around its designed, but non-centroidal rotation axis, and from preventing the rotor from being able to revert to and spin in its natural state about its centroidal axis. This condition is created by the interaction between centrifugal forces from inherent axially distributed mass axis eccentricity, and a countering "torque moment" from the eccentric rotor's overall centroidal moment of inertia. ("Torque moment" refers to the constant net radial moment applied during angular acceleration within a forced non-centroidal rotation, created between the forced rotation centerline and the eccentric center of mass at a given radial plane that would tend to rotate or "flip" the rotor about its longitudinal centroidal mass axis). The resultant journal motion is within the bearing clearances, while the rotor mass axis remains in a "rigid mode" state.

This is contrary to the general assumption in rotor dynamic theory used in rotating machine design and in industry that views "unbalance" akin to the typical mathematical model as being a force vector from an equivalent point mass, acting radially out from the journal axis (as a non-rotating inertial reference frame), exciting the rotor's first mode harmonic at the "first critical speed". Note that "rigid mode" here can incorporate quasi-static bending/deflection as an analogous eigenvector of the rotor body, with that deflected shape maintained while in synchronous precession, but it does not include cyclic-flexing of the rotor body in a linear harmonic oscillation of a mode shape (as in a plucked resonating guitar string), notwithstanding any ellipticity of the orbit. Linear harmonic oscillation does occur but only within the bearing supports, not in the rotor body itself.

2.2. ***Speed thresholds of rotor behavior.*** For a stable operating rotor, the operating speed/frequency of the rotor in a rotating machine must be lower than the rotor's natural harmonic resonance frequency in an unconstrained free state. That implies that the rotor should behave as a rigid body within its operating speed range. When rotor operating speed is equal or slightly higher than the rotor's true natural harmonic resonance frequency (flexural mode), the rotor becomes highly susceptible to instability at such speed, when it is excited by any external disturbing forces (light rub, aerodynamic flow surge, asymmetric torque, pulsating torque, etc.).

Cylindrical rotors in rotational motion in horizontal rotor-bearing systems in a gravity environment must remain sufficiently functionally "rigid" in relation to linearly oscillating flexible supports (bearings, oil, pedestal, etc) in order to maintain stability in operation. A "rigid" rotor in this case means that at system sub-critical, critical and super-critical operating velocity ranges, the rotor itself will not be excited in its first inherent fundamental harmonic flexural mode, meaning the rotor body itself must not reach a condition of fundamental resonance or harmonic linear oscillation (although the rotor-bearing system, relative to global coordinates, may indicate such linear oscillation in the supports). This linear oscillating motion of support springs is orthogonal to the static deflection of rotor longitudinal "springs" in longitudinal strain. The excitation of the linear support "springs" results from a forced rotation, in a gravity environment, of the rotor mass axis synchronously precessing around the journal centerline axis.

Overall, there are three key speed thresholds that can dictate the regime of observed rotor behavior and stability (Figure 1). One, the first *system* critical velocity governs the transition between forced non-centroidal rotation and precession of the mass axis about the journal axis, to a whirling of the journal axis about the mass axis. Below this region, bearing journal stability is gravity-governed, while above this region, stability is a combination of inertial self-centering and hydrodynamic oil forces, which depends on sufficient gravity load and bearing and support design. Two, the speed of twice the first system critical speed marks a point in which a response of an apparent subsynchronous whirling is easily excited (observed to occur at the speed of the first system critical, as a half-waveform eigenvector that "looks like" a first harmonic response). This speed may also excite subsynchronous whirling during coast down when rotor angular velocity coincides with 2x the first system critical speed. Three, the natural harmonic resonance frequency of the "free" rotor marks the key speed above which any small excitation can initiate synchronous internal bending/flexing of the rotor body between the bearings, which is not controllable by bearing damping or rotor self-alignment.

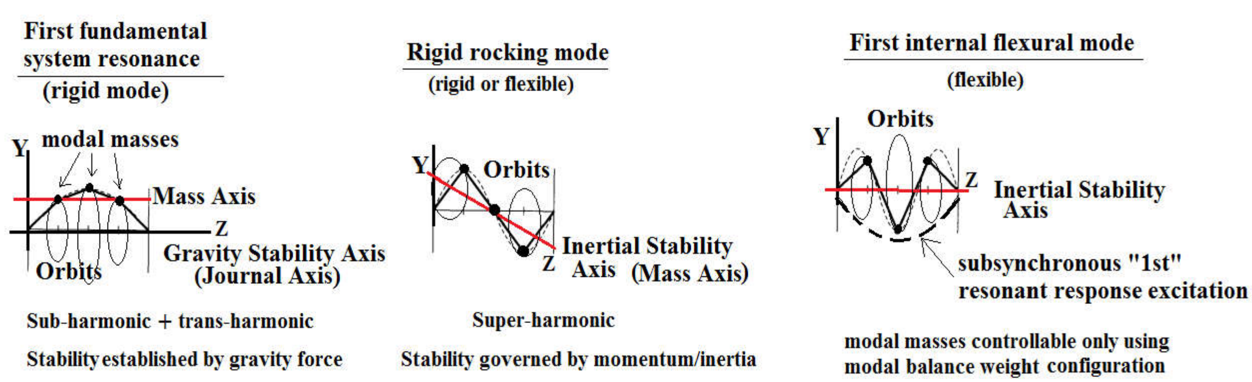


Figure1. Rotor response modes, orbits and stability axis with increasing speed.

2.3. ***Rotor stability and shaft centerline motion.*** The gravity-dependent pseudo-static stability of the journal axis and its position in space (within bearing clearances), up to and at the first system critical speed, is established by the interaction between the apparent gravity load (the rotor's attitude angle, as the resultant from real gravity force and orthogonal reactive lateral force) and the friction force of the flowing pressurized bearing oil in reaction to the surface of the spinning journals as it passes through the "orifice" at minimum oil film thickness (at the point/position of the apparent gravity load). The end of this region corresponds to the angular velocity with a phase angle reaching 180 degrees, where there is a narrow window of operation in natural stability (Figure 2). This is a defined point in space and time when the rotor reaches a state of ideal dynamic self-balancing and self-centering of the centroidal mass axis. This is also a transition point passing from stability and shaft centerline location governed by gravity alone to the more dominant inertial self-centering governing the pseudo-static state of the shaft centerline (SCL).

The further self-centering and orientation "in space" of the SCL at super-critical velocities, relative to global fixed (Newtonian) coordinates, is proportional to rotor drive torque and is governed by the axially distributed moments in each of the two fundamental-mode modal elements (comprising each axial half of the rotor). Each of these modal elements is rigidly constrained at the rotor's overall axial center of mass (COM) on its one end (shared by both modal elements), and elastically constrained at the outer ends at the bearings and supports.

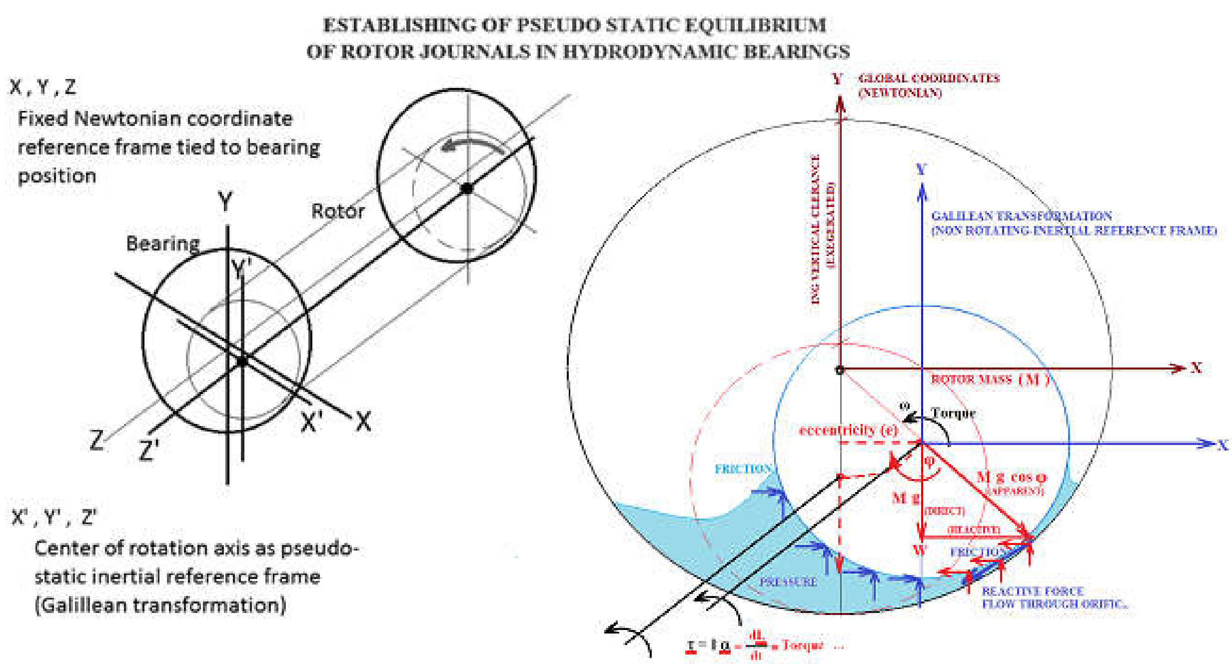


Figure2. Pseudo-static stable position of the rotor journal in hydrodynamic bearings

2.4. ***Transition through the system critical speed range.*** Upon crossing the critical velocity peak (at a phase angle of ~90 degrees), the rotor body centroidal axis reverts to "natural motion" (Newton's 1st Law of Motion) and the centroidal mass axis itself becomes the non-rotating reference frame in place of the journal axis as had been up to this point, with the centroidal axis becoming rigidly constrained in space at a nodal point at the rotor's axial center of mass. While displacement orbits at the bearings appear identical across this transition at the critical speed, both before ~90 degrees and after ~ 90 degrees phase angle, the centers of the orbit changes relative to external fixed global coordinates (what the fixed measuring probes represent), and observed journal displacements decrease with further rotor acceleration from the system critical peak until reaching a phase angle of 180 degrees, with the rotor at its new, inertially self-centered pseudo-static position.

During acceleration of a flexible rotor from the beginning of its system critical velocity range, and during a phase shift from 0 to ~ 90 degrees, the rotor is undergoing pseudo-static bending/deformation due to the conversion of external energy to rotor circular motion, in combination with internal radial asymmetric stress and axial strain adding to the rotor's internal potential energy. When the rotor reaches a phase angle greater than ~ 90 degrees, and after a switch of its non-rotating reference frame as its center of whirling rotation, the stored potential energy from the deflected/stretched rotor "spring" is returned into the system in the form of kinetic energy, until the energy is dissipated during further rotor acceleration, and the self-centering effect governed by inertial forces completes, defining the end of the critical velocity range at 180 degrees phase shift.

This entire "self-centering" process is akin to what occurs in a vertical rotor, even if eccentric, as a vertical rotor would naturally spin about its centroidal mass axis nearly immediately upon the application of drive torque (as bearing support stiffness perpendicular to gravity is minimal, and therefore so is the *system* stiffness and the first "system" critical speed). By contrast, in a horizontal machine with journal surfaces held down by gravity to force rotation about a non-centroidal journal axis, sufficient rotation speed and momentum must build before this transition can occur across the first system critical speed. Similarly, a horizontal rotor on a soft-bearing balancing machine (where balancing is done at a speed above the *system* 1st critical speed), with a single degree of freedom in the horizontal plane with very low support stiffness, also quickly self-aligns its rotation to its mass centroidal axis at low speed, relative to a fixed/Newtonian coordinate reference to which the rotor was installed and aligned in its state of rest on the balancing machine. This change of non-rotating reference frame position can be observed on a shaft centerline plot. (Further, this is the reason why the balancing of a bowed rotor or a rotor with a significant mass axis eccentricity on low-speed balancing machines with soft supports, using the standard static-couple method of balancing, balances the rotor about its mass axis, not its journal/coupling centerline axis, often leading to high vibrations when coupled to another rotor in an operating machine.)

2.6. ***Above the first system critical velocity*** With further acceleration above the first system critical velocity region, the rotor mass axis continues self-centering, governed by inertia forces from net residual axially asymmetrically distributed eccentricities, proportional to angular velocity until reaching maximum operating speed. In installed operating machines, the residual eccentricities can be internal to the rotor and external to any change in alignment of the bearings, or can be due to discontinuity of the mass axis across coupled rotors at the rigid coupling connections to adjacent rotors in a rotor train. Since this self-centering is driven by torque, self-centering will also continue similarly through the machine's full load range as torque increases, observed as a drift in the shaft centerline position.

Above the first system critical speed region, with the rotor body mass axis as a pseudo-static, self-centering, non-rotating inertial reference frame in space (within clearances of the bearings and seals), with further acceleration, the journal geometric centerline at each rotor end continues in conical synchronous whirling around the mass axis, with journal orbit amplitudes proportional to rotor angular velocity (assuming the rotor perpendicular to gravity, and now inertially constrained via a nodal pivoting point at the axial center of mass). The interpretation of the sensor data at the designated measuring points coming from the sensors referenced to global coordinates creates a perception of a rotor rocking motion "out of phase" as though the rocking were about the same journal axis centerline reference as seen below the first system critical. In reality, the total rotor is pivoting at the axial center of the mass axis with the mass axis self-centered/aligned in space, and the journals at each end are whirling about the mass axis (within bearing and seal clearances) as two opposing cantilevers from the axial center of mass. This rocking condition is not a second harmonic rotor response but arises as a manifestation of the overall rotor mass eccentricity magnitude being axially biased relative to the rotor's axial center of mass. The axial side to which the eccentricity is biased becomes a driving end, and since the rotor has some rigidity, it will drive the other end “out of phase” across the center of mass, while the outboard ends remain conically whirling.

A "second system critical" response can also occur superimposed on top of this rocking motion, and depends on the axial asymmetry of the eccentricity distribution and its magnitude. This second system critical represents a process self-similar to what occurs within the first system speed region of the total rotor, but now acting on each cantilevered whirling half of the rotor. The rotor half with the greater residual eccentricity similarly deflects from the corresponding reactive centrifugal force, reacting against the constraints at the bearing and at the nodal pivot point at the total rotor center of mass, and progresses through another 180 degrees of shift in phase angle, again self-centering about the mass axis as now governed by the residual eccentricity on this cantilevered half of the rotor. Nonetheless, any "second system critical" response should be balanced as a part of the solution (and correct axial distribution) of the first system critical response, since by removing the source of excitation, no further response amplitudes should be seen at all at higher speeds.

In summary, the states and reference frames of a rotor as it is accelerated from a state of rest through the system fundamental critical velocity and to operating speed are presented in Figures 3a, b, c, d and e.

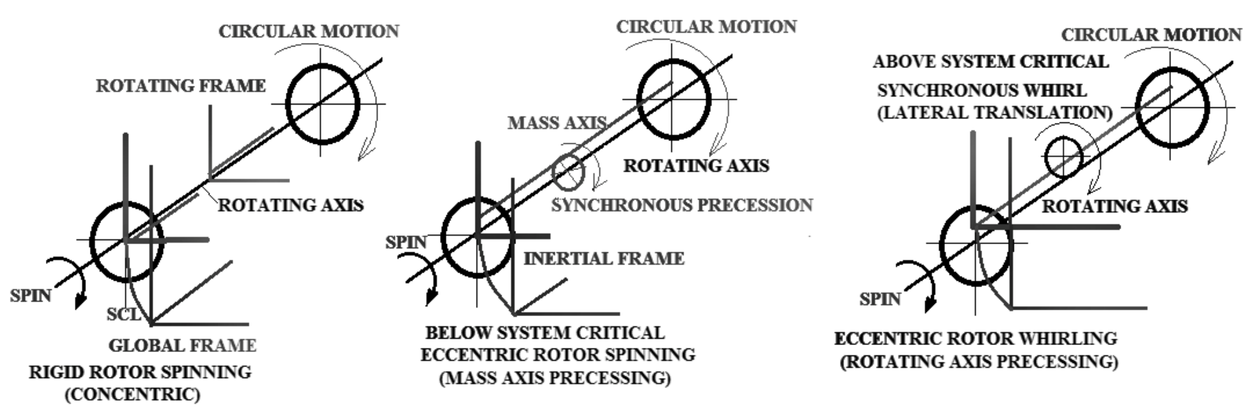


Figure 3a Concentric rotor, **Figure 3b** Mass axis precessing **Figure 3c** Journal axis whirling

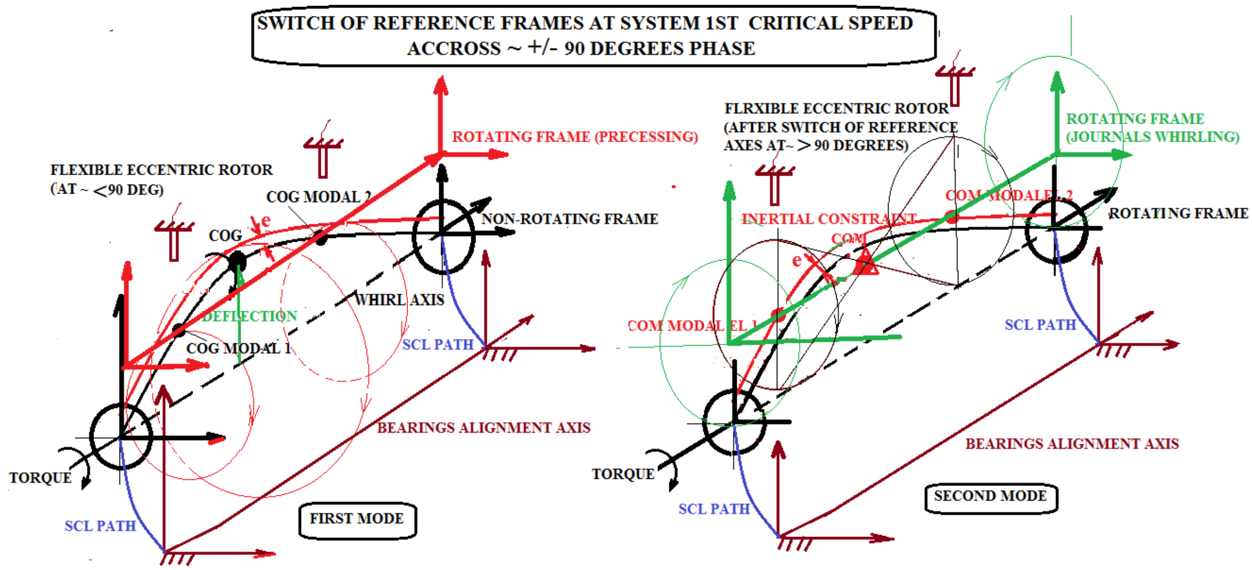


Figure 3d Flexible rotor mass axis precessing, **Figure3e** Flexible rotor with journals pivoting.

3. Application to rotor balancing and design

Considering the difference between a continuous rotor's natural fundamental harmonic resonance response in a free state, and the forced pseudo-static deformation and self-centering transition that occurs through the system first critical speed region, it is then better understood why it is crucial to restore radial mass symmetry about the designed journal axis. It follows that large rotors with an eccentric mass axis, whether bowed, "rigid" or "flexible", must be balanced by compensating and mirroring the mass axis with dynamic forces from balance weights, preventing the quasi-static deflection precession response and the associated switch in axis of rotation and mass axis self-centering.

To simplify the visualization of the mass axis in a rotor with a continuous axially distributed eccentricity, the eccentricity can be reduced by superposition in a minimum of three axial planes along on the rotor's longitudinal axis: at the total rotor center of mass, and two points at the centers of masses of the modal elements of the rotor fundamental harmonic mode (each half of the rotor). The use of a minimum of three planes derives from the formula 2N+1, with N being the ordinal number of the mode of the analogous eigenvector being viewed. 2N+1 provides the minimum number of divisions of the rotor (eigenvector peaks and nodes) where there is no bending between divisions, and each resulting modal element behaves as fully "rigid" through the associated speed range. Therefore, the mass axis can be accurately represented by reducing the distributed rotor eccentricity to its total amount and angle within each "rigid" modal element, and placing that superposition of eccentric mass at the respective center of mass of each of these "rigid modal elements". The principle of dividing a rotor into modal elements and associated locations of weight placement is shown in Figure 4.

These points of superposition provide a practical guide for rigid-mode balancing by dynamically compensating the centroidal mass axis, but in a distribution that is then assured to not otherwise bend or deform the rotor from its intrinsic shape. For eliminating the response at the first system critical (where 2N+1 gives us three planes, and two modal elements), a distribution in three planes mirrors the mass axis about the journal axis by utilizing three, 180-degree opposing counter masses, all laying on the same line and axially proportionally distributed to match the eccentricity distribution. Weight distribution can be up to 100% on one or the other side axially, but never with more than 50% of the total correction weight at the total rotor center of mass at the mid plane. In a typical large turbine rotor, the same affect is still correctly achieved using the rotor mid plane and two end planes, even though the end planes do not align with the modal element centers of mass. (A large flexible rotor with significant eccentricity can never be properly balanced if only two planes are present.) In the case of an eccentric but torsionally twisted or spiral rotor mass axis, correction should be done still in three planes in the same manner, but first broken into orthogonal components, creating a 3-plane solution only in the “X” direction, and then only in the “Y” direction, following the respective sensors measuring the rotor motion.

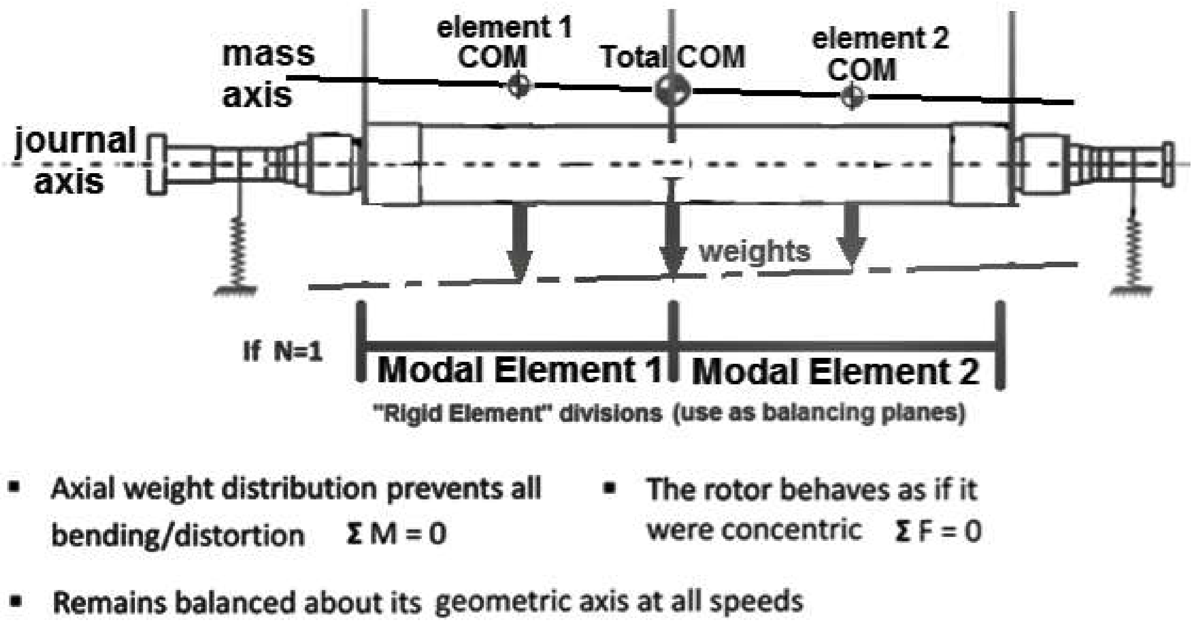


Figure 4 Graphical presentation of the division of rotor modal elements, and weight placement distributed in three balancing planes to resolve a first system critical response and to mirror the eccentricity distribution and mass axis.

In all cases, the balance solution is best determined at angular velocities up to the system first critical velocity corresponding to a response phase angle just below ~90 degrees, where the measured amplitudes are near a maximum. This ensures that the rotor rotation is still forced-centered about its journal axis when the response is measured, and assures that the balance solution directly compensates the effects of mass eccentricity about the rotor's journal axis.

At higher speeds, after satisfactorily resolving the first system critical response, any out-of-phase response amplitudes measured above the first system critical speed should not be considered as a couple unbalance or "dynamic" unbalance, as decomposition along with a "static" unbalance of some total mass distribution. The same unbalance, which is all on one radial side of the rotor (like a bow, but with axially asymmetric distribution) is causing both the lateral response and "rocking" response, and should be resolved as combined in a single balance solution, of which the *axial distribution* of correction masses (in a minimum of three planes) is the key factor. Although the resulting motion above the first system critical velocity region may appear "out-of-phase", the forces causing it are not. As such, this conical whirling condition should not be "balanced" with a couple shot at the journals. A couple shot does not actually restore symmetry about the journal axis, which must be the primary goal. Recalling that rotor rotation is now self-aligned to and centroidal about the mass axis, a calculated couple shot only "bends back" the journals at the outboard rotor ends relative to rotor rotation about the mass axis.

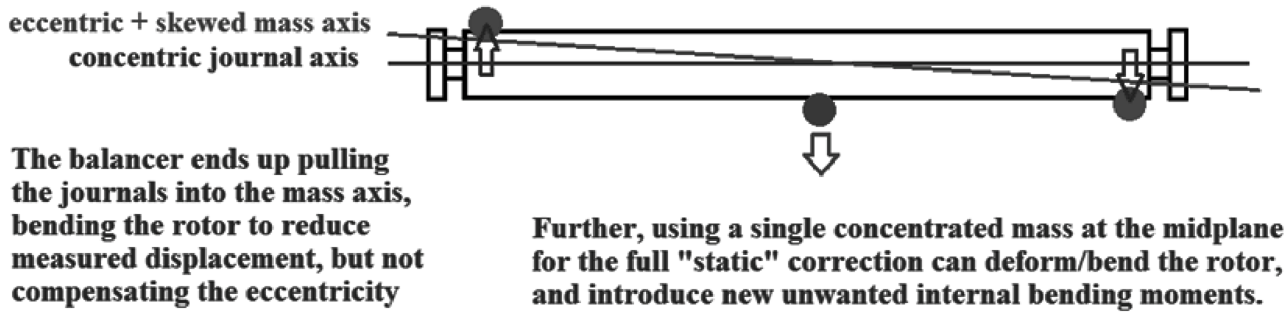


Figure 5 Applying a standard "static + couple" balance solution to attempt to resolve out-of-phase motion above the first system critical, leaving the rotor "balanced" about its mass axis.

Rather, balancing the out-of-phase motion should be done with a half-couple, at the total center of mass and at one journal (a modified "s"-shot), or at the total rotor center of mass and at a balancing plane close to the center of mass of the modal element(s) that comprise each half of the rotor. This modified "s-shot" causes no net change in the already-resolved and shifted mass axis position after having suitably corrected the first system critical response, but is effectively axially redistributing the weights for the first critical correction. Weight should be placed first on the side of the rotor that in a polar plot shows both "loops" of the response from the first and second system critical speed regions in the same general phase angle direction, since this identifies it as the driving end with greater mass eccentricity (the distribution of eccentricity is axially biased toward this side). This should be done at a speed above the first system critical as a trim shot only if an initial solution in three planes is not yet properly axially distributed (the balance solution determined from a speed just below 90 degrees of the first system critical).

It could be noted that the first system critical amplitudes can be reduced even with standard methods of a "static and couple", but typically when placing the "static" shot in one plane, the rocking mode or second system critical response will not improve or will be increased. Likewise, reducing the measured rocking mode amplitude can also be achieved by standard methods of a "couple shot" from influence coefficients, but then very likely the first system critical response will be increased. Frequently in a balancing facility using standard methods with an eccentric, flexible rotor, there is a continuous back-and-forth between improving one response but worsening the other. However, by simply radially and axially compensating the mass axis of flexible rotor in three planes, it resolves all responses at the same time and for all speeds. If there is effectively no eccentricity and the journal and mass axes are coincident, then there will be no vibration responses at the first system critical speed or higher, and the rotor will maintain natural operation about its installed and coupled journal axis at all speeds after being installed in an operating machine. As a secondary benefit, this overall approach is more economically advantageous to be able to balance primarily at lower speeds, saving drive power and time in the balancing facility.

3.1. ***Resolving vibration on smaller, higher speed machines.*** Although the same rotor dynamic processes described previously apply to all rotating machines, there is a distinction in solving the effects of rotor eccentricity or unbalance on high-speed rotating machines like compressors, jet engines or turbochargers, versus on large, low speed turbine-generator sets in power generation. Traditionally, vibrations and stability issues on high-speed rotating machinery with a high power density ratio, and with sufficient flexibility of modal elements, are resolved by a passive approach, primarily by various design modifications of bearings, seals and supports, which effectively account for and allow the self-centering of the rotor to its centroidal axis upon which it can run stably, without being constrained from this "natural" motion. When it is not physically possible to remove all eccentricity by machining or balancing, due to the very small tolerances on smaller high-speed rotors, the key is to accommodate the self-centering of the rotor about its mass axis by sufficiently reducing stiffness in the supports relative to the rotor stiffness. If the bearings cannot be made with sufficiently low stiffness, then another soft support under the bearings can achieve a similar result, allowing the rotor to pull the bearings with it as it self-aligns above the system critical velocity. This approach allows for stable operation at higher speeds, although stability is still limited to an operating frequency less than the fundamental natural resonance frequency of the free unconstrained rotor body alone.

For large turbine-generators with enormous mass and inertia, this passive approach is less practical and highly dependent on manufacturer design philosophy, and the focus must be instead on recognizing mass axis eccentricity and compensating it to restore symmetry about the journal axis, optimally by applying a balancing correction in three planes while at rotor sub-critical velocity.

3.2. ***In-situ field balancing.*** Various balancing methods based on "harmonic modal" responses at the 1st critical speed, and influence coefficient methods at speeds above the first system critical speed, are currently broadly used in the service industry when field-balancing turbine and generator rotors in-situ. Most of these methods have been in use for years as a go-to remedy for resolving vibration on an operating machine. However, it is important to remember that the in-situ balancing of assembled operating turbine-generator sets is not truly "balancing", since without being able to properly axially compensate for all mass eccentricity in the required minimum number of balancing planes, the energy "stolen" by the inertia of the uncompensated eccentric mass still remains in the system. As such, in-situ balancing is rather a political and economic compromise by the user, and a physical compromise between a reduction of vibration displacements of the journal axis relative to the centroidal mass axis (at points of measurements), which was expressed as kinetic energy, and the conversion of that same energy into an increase in rotor internal potential energy in the form of increased rotor cyclic stress and internal cyclic bending moments.

3.3. ***Balance quality based on shaft centerline.*** The rotor mass axis pseudo-static self-centering within bearings can be only observed on a shaft centerline plot, and on eccentric rotors is often seen as a continuing drift of the SCL, often in a "wrong" or an unexpected direction and magnitude. This makes the SCL plot an indispensible part of identifying the existence of and the proper compensation of mass eccentricity in a rotor body, or for identifying any external disturbing force influencing the self-centering stability axis in operation.

Balance quality should therefore be additionally quantified by DC shaft centerline data, to carry as much importance as AC amplitude measurements. Any ideally concentric rotor (on a balancing machine, and when well aligned and coupled in a rotor train), should not notably shift its shaft centerline in space once reaching a stable attitude angle in its bearings, and this should correspond to the rotor position at 180 degrees of phase angle just at the end of the first system critical speed region. For a well-balanced rotor, further acceleration should present minimal continued SCL self centering drift with speed (due to unavoidable residual eccentricities). Any continuous drifting of the SCL of one side of the rotor with increasing speed above the first system critical indicates that mass eccentricity is not properly compensated, or bearing center position is changing relative to global coordinates (loose bearing), even if measured displacement amplitudes alone are within the permissible tolerance accepted in industry. Similarly, if significant SCL hysteresis is observed between rotor acceleration and deceleration, it is indicative of unresolved mass axis eccentricity, and the rotor is not truly balanced with restored symmetry about its journal axis. Such a rotor will often present high vibration amplitudes and/or bearing forces once installed in the field when its journal centerline axis (but not its effective mass axis, about which it was "balanced") is aligned relative to the centerline of the bearings, referenced to global coordinates.

3.4 *Rotor runout evaluation.* Balancing is only one part of the overall process of eliminating the effect of eccentricity, and as such, rotor balancing should not be a procedural island for reducing machine vibrations, but for the most effective results in practice it must be directed as a single part of a larger shop process of identification and treatment of rotor mass eccentricity.
Runout evaluation for example is essential to precede balancing activity for among other reasons, to identify if slow roll runout measured on a balancing machine is truly from surface variation (in which case runout subtraction can be appropriate), or if it is a part of the distributed eccentricity/bow that must be compensated, in which case runout subtraction provides a wholly incorrect implication.

4. Alternative look at rotor stability and operating design speed

Before looking more deeply at the rotor dynamics specifically within the system resonance behavior through the fundamental system critical velocity region, it is important to consider rotor stability and instability in its bearings in a more general sense. It is typically considered that instability such as subsynchronous whirling or oil whirl or oil whip is the fault of bearing design or oil quality problems. It is true that proper bearing design is important to be able to absorb the forces that create instability, and bearing design is the key avenue of vibration resolution for smaller and higher-speed machines. However, the root of instability is most often in the rotor itself, originating in the asymmetric forces resulting from unresolved mass eccentricity, and/or from operating a rotor at high speeds above its "free" fundamental harmonic resonance frequency flexural mode. Instability can also be created in combination with insufficient gravity load on a bearing in an assembled rotor train, which allows the asymmetric internal forces from the rotor eccentricity to overcome the stabilizing forces provided by the bearing/oil. (In such cases, vertically raising the unstable bearing can often resolve the problem. Excessively preloading a bearing as a matter of design to prevent instability is not recommended, however.)

Rotating machines are generally designed and modeled on the **assumed** equivalency of circular and linear motions and corresponding Newton's laws, and this has led to a common mixing of linear harmonic oscillating resonance and rotating system linear resonance behavior. It is important to always distinguish these as related but separate phenomena governed by different factors. This recognition can point to a number of general design guidelines with regard to rotor and bearing stiffnesses and natural frequencies, and the operating speed of the machine, that should be adhered to in order to prevent instability in machine operation (or utilized to diagnose the root cause of observed instability when the rotating system comes outside of the guidelines).

4.1 *Unconstrained rotor natural harmonic versus constrained rotor system critical speed*

Continuous rotors, as machine elements of finite length, must be sufficiently "rigid" and stable in system sub-critical, critical and super-critical design operating velocity ranges to ensure pseudo-static stable operation (a stable inertial reference frame). With that in mind, it must be ensured that any rotor is designed to behave as internally non-oscillating and "rigid" within the machine design operating speed range and up to maximum system external torque, while maintaining energy conversion efficiency. This is dependent on rotor material properties and rigidity, with the rotor constrained by gravity on linearly oscillating flexible supports of appropriate stiffness.

As an initial rule, it is crucial for reliable system dynamic stability that the rotor operates at speeds below the equivalent "free" fundamental harmonic frequency of the rotor body alone. It is also important that the bearing/supports have a lower net stiffness than the rotor body. If a rotor operates at a speed above its "free" fundamental harmonic frequency, it is prone to uncontrolled internal flexural bending that can induce dynamic instability. If the bearing/supports are stiffer than the rotor and the rotor is effectively "clamped" at its ends, this can inhibit the transition from non-centroidal to centroidal rotation (or increase the speed at which it occurs), such that the asymmetric forces from eccentricity within the rotor, which grow with the square of speed, become excessively damaging to the rotor material (or are transferred to the bearing material) before such a transition can occur (marked by the processes within the system fundamental critical velocity range).

The rotor-bearing system with hydrodynamic oil bearings should be designed such that the first system critical speed is lower than 0.5x of the designed operating speed, and operating speed must be less and no more than equal to natural resonance frequency in free-free state. If system parameters are outside these limits, then to build a stably operating rotor-bearing system for a necessary rotation speed requires stiffening the rotor or supports, or softening the supports, changing rotor material or bearings span length, and so on. Ultimately, the maximum rotor operating speed and maximum efficiency of a rotating machine is solely dependent on rotor material properties its fundamental harmonic resonance frequency in a free-free state and bearings and supports stiffness.

As a design rule, the first system critical should be 10-20% below half of the machine running speed. Although, on very flexible rotors with a high L/D ratio of 7 or larger, the critical speed can be much lower. On some generator rotors operating at 3600 rpm, the system 1st critical speed can be as low as ~700 rpm. However, this condition also requires very soft supports to assure that rotor still remains operating in its "rigid" mode**.**

With this, it ensures for the sake of stability that the rotor-bearing system can pass through its first system critical and reach a self-aligned state about its mass axis, including the effects of both radial asymmetry and axial asymmetry. If support stiffness is such that a rotor passes its second system critical speed, it approaches or it gets higher than rotor natural fundamental harmonic frequency of flexural mode, with increasing torque power, operating in this condition can leave the rotor susceptible to subsynchronous instability, especially in conditions where the gravity load on a bearing within a rotor train is reduced. On such rotors it is especially important that mass axis eccentricity is minimized and the rotor properly balanced in three planes. To enhance the likelihood of maintaining stability, even if encountering conditions with reduced bearing gravity load, the rotor should be operating at a speed greater than twice its first system critical speed, but lower than rotor fundamental harmonic resonance in free-free state. In practice on rotor-bearing systems, the second system critical velocity/frequency is not a direct multiple of the first as in the pure harmonic vibration of a string, or perfectly symmetric cylinder, but as a general design parameter for all cases it is beneficial to maintain a factor of 0.5 x or lower between the first system critical velocity and running speed.

In situations , when supports stiffness is changed intentionally by design ,or unintentional due to operating conditions it is important to compare the new system fundamental critical frequency relative to the operating speed. A system (particularly one with stiff bearings and supports, such as ball bearings in rigid housing) that may have originally operated below its first system resonance frequency ("1stcritical"), "self-centering of its mass axis in "space", might be altered at running speeds above its first system critical where the switch in rotation axis to the mass axis occurs. If any axially asymmetric mass eccentricity remains in the flexible rotor, damaging forces in bearings or cyclic bending forces in rotor, from kinetic energy can arise proportional to square of speed. If the bearings supports are not appropriately softened and end up "clamping" the rotor at journals, preventing it from switching its rotation center and its natural reorientation, the result is increasingly large forces that will damage the bearings after a short duration of operation.

Note to eliminate confusion:

1 Rigid rotor in machine with rigid bearings and supports, the system 1^{st} critical speed must be higher than operating speed.

2. Flexible rotor in machine with hydrodynamic bearings and elastic supports, the system 1^{st} critical speed must be 0.5 x or lower than operating speed.

Further, significantly reduced bearing load on multiple rotor machine rotor train can effectively increase the length L of the rotor span between supports, acting to reduce the combined system resonance frequency, and also reduce the effective "free" resonance of this longer span (changing the ratio of K/M). This can potentially shift the free resonance frequency of this coupled rotor train below the operating speed, allowing for an internal flexural resonant response in the rotor train, with the associated resulting motion and forces inducing journal subsynchronous instability. Similarly, when bearing is loose it reduces stiffness and increases susceptibility to rotor instability.

4.2 *Relation between harmonic resonance, critical speed, and support stiffness*

The following symbolic relations were developed as a summary and design guideline of the key factors that affect the critical velocity of an accelerated rotor-bearing system that govern the stability of the rotor in its bearings. These relations are not intended as an equation for calculating frequency values, but as a simple tool to recognize the specific sources of potential rotor instability and sensitivity to "unbalance" to set limits to the relevant values for optimum design (regarding rotor material properties and stiffness, bearing stiffness, and maximum stable operating speed and torque), and to identify the system properties that can cause observed subsynchronous vibration or high sensitivity to unbalance. A main challenge in assessing these parameters comes from the fact that rotor fundamental harmonic resonance in free-free state, and a linear system response at 1st critical speed, are not the same physical phenomena. It is impossible to mathematically combine these parameters into a single comprehensive equation, since they represent a combination of forces and affects in multiple reference frames, where some terms represent angular vectors of rotor internal mass, and linear vectors of system springs in gravity environment, such the terms can only be expressed as proportionality. Therefore, these relations should be interpreted more generally as flowchart of parameters of comparison.

First, it is helpful to distinguish differences between equation generally utilized for determining fundamental harmonic natural frequency as a rotor **linear** oscillation, independent of distance L between nodes, and equation utilized for determination of fundamental harmonic resonance frequency of continuous rotor as geometric body derived as harmonics in music (e.g. guitar or harp strings).
The well known equation for determining unconstrained rotor fundamental harmonic resonance is generally expressed as:

$$\omega = \sqrt{\frac{K}{M}} \text{ (Hz)}$$

where:
K= rotor stiffness (elasticity), orthogonal to longitudinal axis (lb/in)
M= rotor mass m= W/g (lb sec^2/ in)
ω = fundamental harmonic (1/sec)

The equation is almost exclusively used in rotordynamics for modeling rotors as a point mass on linear viscoelastic supports, and calculating rotor's harmonics based on Euler-Bernoulli beam theory. The equation above represents the rotor first mode as a linearly oscillating harmonic wave. But, this does not actually represent the real accelerated rotor in circular motion, constrained in bearings by gravity on horizontal machines, and its behavior at its first system critical velocity.

When a rotor is considered as a continuous geometric body whether rigid or flexible, and not as a point mass on a spring, and free in a state of rest without gravity influence (such as vertically oriented with the mass axis in the direction of gravity), it still has an inherent free fundamental harmonic resonance frequency. The free harmonic resonance response is excited by kinetic energy as an impulse force to the rotor body perpendicular to the rotor's longitudinal axis, with a resulting initial longitudinal traveling wave response propagating through the rotor material particles and reflecting, with the rotor molecular masses then vibrating internally at all harmonics of the fundamental frequency as internal standing waves. The energy from the impulse acts as a disturbing force and converts to radial stress and axial strain, disturbing the state of equilibrium between the molecules constituting the rotor mass, and electromagnetic restoring forces propagate longitudinal vibration displacement patterns in the axial direction at the velocity V. The initial longitudinal traveling wave in a free rotor would have a velocity determined based on the longitudinal stiffness of the rotor (analogous to tension, as in an oscillating constrained guitar string). The added energy from the impulse perpendicular to mass axis, converts to sound energy returning back to environment at velocity of sound shown in equation below:

$$V = \sqrt{\frac{T}{m / L}} \quad \text{(in/sec)}$$

The fundamental frequency is a standing wave such that the wavelength, W, is twice the length of the rotor, or 2L, and the corresponding eigenvector seen in the rotor is a half-waveform of a sine wave. The rotating machine fundamental resonance frequency at rotor critical velocity could then be written as

$$f_1 = \frac{\sqrt{\frac{T}{m / L}}}{2L} \quad \text{(1/sec)}$$

Therefore, an alternative presentation of the linear free resonance eigenvector of a rotor can be made in the following form, where T represents longitudinal tension and the mass M is also shown broken into its constituents, of volume, density and gravity.

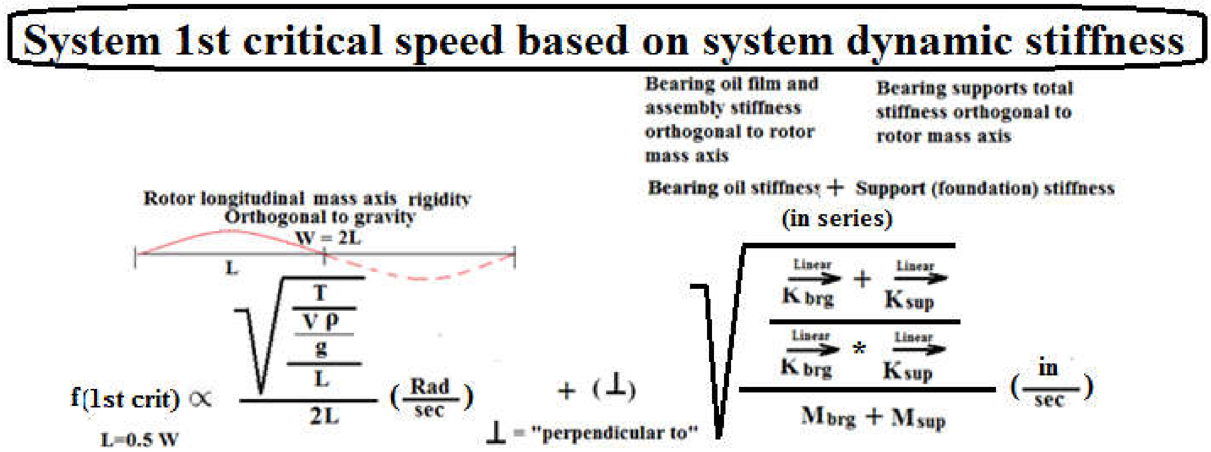
$$\omega_{res} = \frac{\sqrt{\frac{\frac{T \Delta L}{\frac{V \rho}{g}}}{L}}}{2L} \quad \text{(1/sec)}$$

where:

T = longitudinal tension (lb/in2)
Δ L = elongation (non-dimensional)
V = rotor volume (in3)
ρ = Material density (lb/in3)
g = gravity acceleration (in/sec2)
W = frequency spatial wave length)
L = distance between rotor constraints

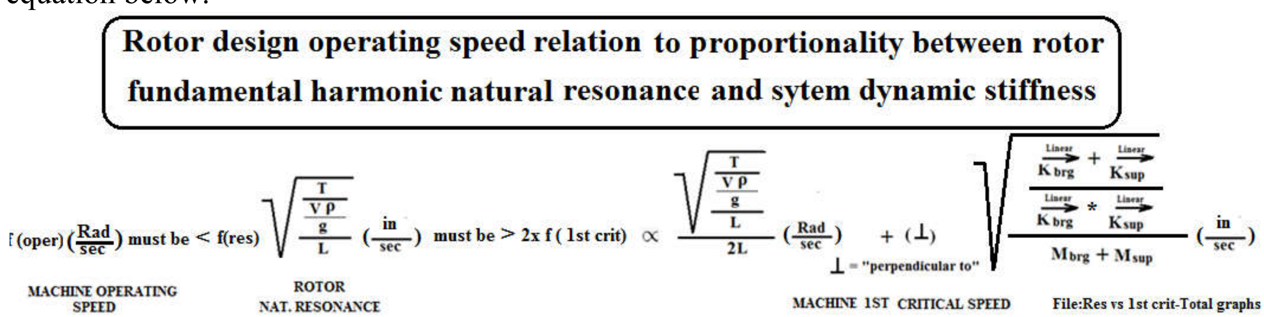
This version allows for the representation of the different types of natural frequencies comprising the overall system critical velocity. The "spring" stiffness, K (lb/inch), in a rotor is typically considered in reference to the bending displacement measured perpendicular to the longitudinal axis of the rotor for a given force applied orthogonally to the rotor's longitudinal axis. Since such a measurement is impractical to execute on large massive rotors, designers calculate the rotor's stiffness based on gravity sag. Such bending displacement also produces longitudinal tension in a constrained rotor.

Since the first system critical of a rotor-bearing system (horizontally mounted, in a gravity environment) must take into account all support stiffnesses, a symbolic representation of the parameters governing the first system critical velocity is shown in the following relation. Note that the frequencies represented by the respective formulas are not truly equivalent, with the rotor frequency representing a longitudinal standing wave, and the supports representing frequencies corresponding to linear oscillation perpendicular to the rotor. The overall system critical speed range is governed by when the spin frequency of the flexible rotor with eccentric mass axis (which is in growing pseudo-static bending/deflection from centrifugal force from eccentricity) approaches the natural resonance frequency of the supports (as linear oscillation), which creates the measured rise in total precession translational orbit amplitude of the rotor in space, within bearing clearances.



Returning to the previous relations between rotor free natural frequency, constrained rotor natural frequency, and system fundamental critical velocity, the following comparative relations can be presented as general guidelines for the relation between various parameters described previously. As a simple summary, a first system critical response “seen” by sensors mounted relative to fixed (Newtonian)coordinates, simulate the rotor as a half-waveform eigenvector, and the second system critical sees the rotor as a full-waveform eigenvector. However, both of these are "rigid" and in pseudo-static deflection. A flexible rotor should not operate at a speed that produces any higher modal eigenvectors beyond these, because it can initiate rotor internal flexural bending/oscillation at the fundamental natural frequency of the rotor alone, which can create instability such as oil whirl during operation. The objective of designer is to maintain rotor as a rigid body, by properly tuning support stiffnesses.

The relations and dependency between rotor fundamental harmonic resonance, rotating machine stable operating speed and operating rotating machine system total stiffnesses, are shown in equation below.



5. Resonance effect, and critical and resonance velocities

The dynamics of a Jeffcott rotor used in the study in this paper, as disc rotating on a massless shaft, comprises the theoretical basis of many facets of rotor dynamics that have been studied in various scientific directions. It provides the understanding of physics for the processes that take place during a change in rotor angular velocity. The equations of Jeffcott rotor dynamics were obtained by the methods of classical mechanics in rotating and stationary systems of coordinates [1], the Lagrange analytical method of integration in time [2], and also including the Lagrange method to account for the force of external attenuation [3]. The obtained equations were based on the spin of the rotor about its center of mass while simultaneously having the center of mass whirling about the rotor's journal centerline. The equations described the situation where the direction of any force acting to bend/deflect the rotor and the direction of the resulting bend/deflection were always collinear. Rotor resonance was considered as the process in which unlimited bending/deflection of the shaft would occur, along with a simultaneous change in position of the center of mass relative to the centerline of rotation/whirling and the geometric center of the disc. It was considered that this change in position of the center of mass occurs instantly upon the rotor reaching its critical velocity at approximately 90 degrees phase angle referenced to fixed (Newtonian) coordinates. After, with further acceleration, the super-critical mode of rotation is reached in which self-centering of the rotor occurs at the end of the system 1st critical speed range and a phase angle of 180 degrees. The introduction of the concept of viscous friction into the dynamics of the rotor [3] allowed for the presentation of the resonance phenomenon and rotor self-centering as occurring during a process that has a specific duration. At the same time, the resonance phenomenon had remained defined by only the critical velocity calculated as a single speed. However, experimental evidence has shown that the resonance phenomenon on flexible rotor/shaft systems starts at a considerably earlier velocity than the point calculated as the critical speed, and likewise the self-centering of the rotor occurs at velocities that exceed the critical velocity.

Experiments showed **(Figures 14**) that resonance phenomena take place within a definite range of velocities that are close to the first system critical velocity of the rotating machine. However the limits of this velocity region were not theoretically determined with sufficient accuracy. Experiment showed that the calculated predicted critical velocity usually does not coincide with the real velocity where the maximum peak deflection of the rotor shaft takes place (as measured relative to global (fixed) coordinates). Inertial theory of rotor dynamics [4, 5] showed that the preliminary preparation of a rotor to transfer through its first system critical velocity starts at sub-critical velocities. This period of initiation of the behavior associated with the critical velocity correlates to the velocity range encompassing a response phase angle shift from 0° to ~90°, during which the rotor's mass centroidal center (or mass axis) is precessing around the geometric center (journal constraints axis). At the same time, the bending/deflection of the shaft does not yet exceed the magnitude of the rotor's eccentricity. A further increase in the bending/deflection of the shaft beyond this threshold indicates the beginning of the resonance phenomena which, in agreement with experimental evidence, governs and ensures the continuation of the shift in phase angle up to 180°.

Note that the main theoretical research of Jeffcott rotor dynamics was conducted in conditions where the "rotor-shaft" system rotates uniformly, at constant velocity, and the necessary rotating moment (drive torque) is created by the external driving gear. This means that a change in rotor velocity was considered as inessential to the model. In practice however, a change of rotor velocity occurs with acceleration of the rotor, and at the same time, the resonance phenomena becomes apparent at any instance when the rotor makes accelerated or decelerated rotation under the effect of torque applied by the drive gear, or under the effect of inertia moments during the rotor deceleration with drive torque stopped. In connection with the aforesaid descriptions, it was stated necessary to define the specific area of rotor velocities that are connected with and define the system's resonance phenomena. For this, it is necessary to determine the rotor velocities when the resonance phenomena of the system are started and ended. Moreover, it is necessary to explain the physics of the resonance phenomenon and mechanism by which the observed phase angle shift occurs from 0° to 180° and during which the rotor's center of rotation (non-rotating reference frames) transfer while passing through the system critical velocity at ~90 degrees phase.

For a continuous rotor as a system in which the mass of the shaft has significant influence on its dynamics, the same methods should be applied to the rotor centroidal mass axis as a line crossing three points, the total rotor center of mass and the two centers of masses of two fundamental harmonic modal elements in null mode which comprise each half of the rotor.

5.1 *System of rotor dynamic equations*

In this section we first see the system of equations that comprehensively describes the uniform rotation of a Jeffcott rotor in conditions of weightlessness and the absence of any external or internal friction that inhibits rotation. The system of equations determines the transfer mode of the rotor's rotation through subcritical velocities as well as the rotor's supercritical rotation mode with supercritical velocities. This system of equations will be consistently referenced while also considering other issues, since the rotation of any accelerated rotor invariably transfers into uniform rotation of the rotor. The scheme of rotation and acting inertial moments is explained in Figure 7 shows the rotation of the rotor around the shaft's geometric centerline and the lateral orbital rotation of the shaft around the journal/support axis. Additionally, the rotating and inertial moments of the "rotor-shaft" system are noted. The rotating moments act around the shaft's geometric axis. The inertial moments are stipulated by the uneven distribution of mass around the shaft's geometric axis and rotational axis. The inertial moments simultaneously act around the shaft's geometric axis and around the journal rotational axis. The location of the rotor as shown in Figure 7 corresponds to rotor rotation with an initial velocity equal to the one when the shaft is bending under the effect of centrifugal force (i.e. centripetal reactive force), analogous to when a continuous rotor-shaft undergoes pseudo-static deformation.

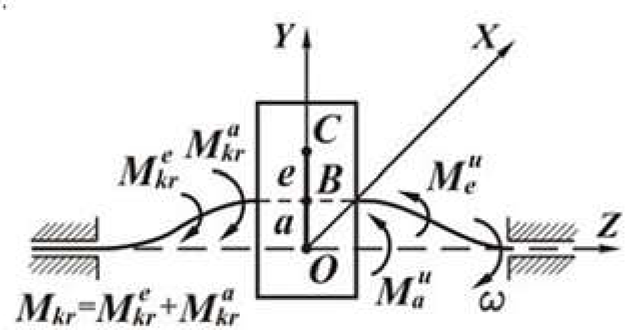


Figure 7

The scheme of forces and moments that act upon the rotating rotor is presented in Figure 8, representing the system at ~ 45 degrees phase shift angle **(β)**.

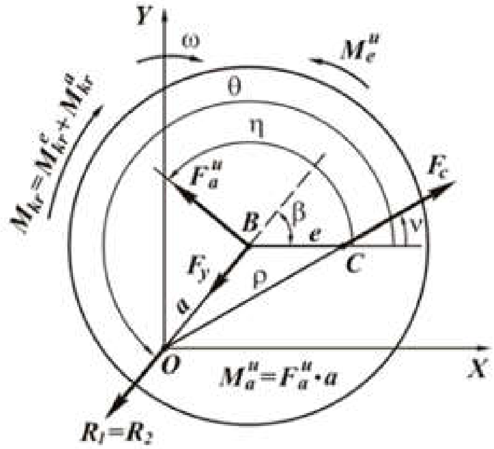


Figure 8

The system of equations that describes the complete dynamics of a Jeffcott rotor in the complete range of velocities takes the following form:

$$
\begin{aligned}
&M_{kr} - m\omega^2 e^2 - m\omega^2 a^2 = 0\\
&M_{kr}^{e} - m\omega^2 e^2 = 0\\
&M_{kr}^{a} - m\omega^2 a^2 = 0\\
&R_1 - \frac{1}{2}ka = 0\\
&\sin\beta = \frac{a}{e}\\
&m\omega^2\left(a + e\cos\beta\right) - ka = 0
\end{aligned}
\qquad (1)
$$

At the same time, the following forces and moments act upon "rotor-supports" system with statically unbalanced rotor depending on the rotation velocity:

$$
\begin{aligned}
&M_{kr} = M_{kr}^{e} + M_{kr}^{a}\\
&M_{kr}^{e} = m\omega^2 e^2\\
&M_{kr}^{a} = m\omega^2 a^2\\
&M_{ob}^{u} = M_{e}^{u} + M_{a}^{u}\\
&M_{e}^{u} = m\omega^2 e^2\\
&M_{a}^{u} = m\omega^2 a^2
\end{aligned}
\qquad (2)
$$

$$F_c = m\omega^2 \rho , \qquad \sin \nu = \frac{a}{\rho} \sin \beta$$

$$\rho = \sqrt{e^2 + a^2 + 2ea\cos\beta}$$

$$F_y = ka \qquad \theta = \beta + 180^\circ$$

$$F_a^u = m\omega^2 a \qquad \eta = \beta + 90^\circ$$

$$R_1 = R_2 = \frac{1}{2}ka \qquad \theta = \beta + 180^\circ$$

Where:

m - rotor mass;

ω - rotor rotation velocity;

ω_r - commencement of subcritical critical velocity range at ~ 0-5 degrees phase

ω_{kr} - critical velocity at ~ 90 degrees phase

ω_r^s - end of critical velocity range, begin of supercritical velocity range,

e - rotor eccentricity;

a - rotor shaft bend (deflection);

k - coefficient (modulus) of rotor elasticity or rigidity

β - lag angle of rotor's response vector around the shaft's geometric axis; angle of phase shift;

M_{kr} - total rotating moment that ensures the rotor's uniform rotation;

M_{kr}^e - rotating moment that ensures uniform rotor rotation around the shaft's journal axis centerline;

M_{kr}^a - rotating moment that ensures uniform rotation of shaft's geometric centerline together with the rotor around the journal axis;

M_{ob}^u - total inertial moment that prevents uniform rotation of "rotor-shaft" system;

M_e^u - inertial moment that prevents uniform rotation of the rotor around shaft's geometric centerline;

M_a^u - inertial moment that prevents uniform rotation of the shaft's geometric centerline together with the rotor around the journal axis;

F_c - centrifugal force applied towards the rotor's center of mass;

ν - angle which is shaped by centrifugal force and the direction of vector of imbalance;

F_y - shaft's elasticity force;

θ - angle that is formed by the shaft's elasticity force and the direction of the vector of imbalance;

F_a^u - inertia force that prevents rotation of shaft's geometric centerline together with the rotor around the journal axis;

η - angle that is formed by the force F_a^u and direction of the vector of imbalance;

$R_1 = R_2$ - force of reaction of the supports;

This system of equations and scheme of forces and moments can be used for the definition of resonance velocities and the range of steady operating velocities of the rotating machine. Under "resonance velocity" we will consider the velocity that determines the beginning and completion of the resonance process when the rotor velocity is changing during acceleration.

5.2 *Resonance velocity of rotor rotation in supercritical rotation mode*

Rotor dynamics usually assumes that the range of operative velocities covers the whole range of supercritical velocities up to the area of auto-oscillations. However the system of equations that describes the uniform rotation of a Jeffcott rotor shows that the range of operative supercritical velocities starts from the velocity that is 1.74 times higher than critical velocity. To explain this let's show some dependencies of relative parameters of rotor rotation from relative velocity on transfer and supercritical mode.

Relative shaft bend from relative velocity on transfer and supercritical mode has the following view:

$$\frac{a}{e} = \frac{\omega^2}{\omega_{kr}^2} \frac{1}{\sqrt{1 - \frac{2\omega^2}{\omega_{kr}^2} + \frac{2\omega^4}{\omega_{kr}^4}}} \quad (3)$$

$$\frac{a}{e} = \frac{\omega^2}{\omega_{kr}^2} \frac{1}{\frac{\omega^2}{\omega_{kr}^2} - 1} \quad (4)$$

These dependencies are shown in **Figure 9.**

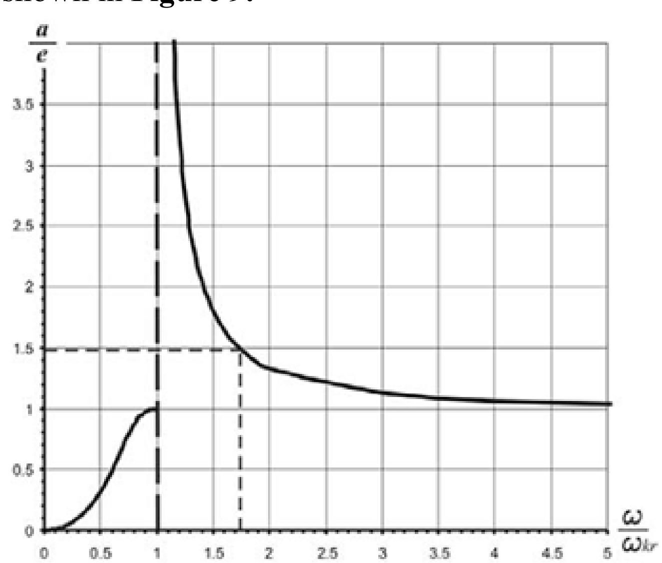


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Figure 9

Let's show the dependency of the relative rotating moment M_{kr}^{a} from relative velocity which has the following view:

$$\frac{M_{kr}^{a}}{ke^2} = \frac{M_{a}^{u}}{ke^2} = \frac{\omega^6}{\omega_{kr}^6} \frac{1}{1 - \frac{2\omega^2}{\omega_{kr}^2} + \frac{2\omega^4}{\omega_{kr}^4}} \qquad (5)$$

$$\frac{M_{kr}^{a}}{ke^2} = \frac{M_{a}^{u}}{ke^2} = \frac{\omega^6}{\omega_{kr}^6} \frac{1}{\left(\frac{\omega^2}{\omega_{kr}^2} - 1\right)^2} \qquad (6)$$

These dependencies are graphically demonstrated on **Figure 10.** As it follows from the chart, on supercritical velocities the relative minimal inertial moment $M_a^u/ke^2 \simeq 6.75$ correspond to the system's rotation velocity, which is approximately ***1.74*** times higher than critical velocity. This velocity is the minimum velocity $\omega = 1.74\ \omega_{kr}$ when uniform rotor's rotation is possible.

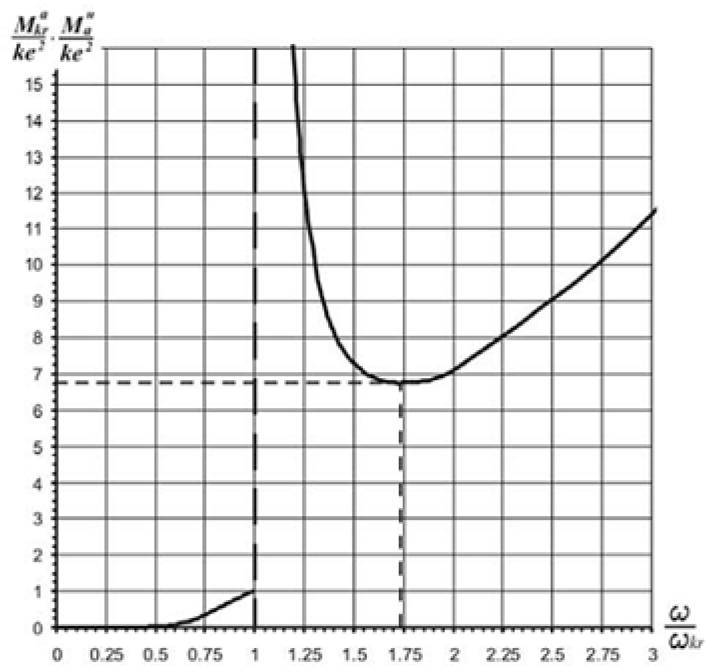

Figure 10

In accordance with the chart **(Fig. 9),** the minimal velocity corresponds to rotor's rotation when shaft's bend is equal to $a = 1.5e$. Such shaft's bend relates to completion of resonance phenomena on supercritical mode during increasing of rotor's velocity.

If $\boldsymbol{M^{a}_{kr}}$ $/ \boldsymbol{ke^{2}} > 6.75$, then velocity of rotor's uniform rotation exceeds the minimal velocity $\omega = 1.74.\omega_{kr}$, **sufficient for transfer of rotating frames at rotating speed corresponding to peak response and at 90 degrees phase angle V (Figure 8).**

If $\boldsymbol{M^{a}_{kr}} / \boldsymbol{ke^{2}} < 6.75$ then the rotation of the system becomes slower
Inertial moment (Fig.10) becomes bigger than rotational moment and we monitor the process never reaches switch of reference frames and rotor's self-centering. I**n classical rotordynamics this would represent a "rigid "rotor in "rigid supports. Operating speed of this type of rotating machines should be always above the system 1st "critical speed".**

If relative rotational moment $\boldsymbol{M^{a}_{kr}} / \boldsymbol{ke^{2}} < 6.75$ is less than the rotational moment $\boldsymbol{M^{u}_{a}} / \boldsymbol{ke^{2}} = 1$ which corresponds to critical velocity $\boldsymbol{\omega_{kr}}$, then the rotor's velocity reaches one of the velocities of transfer mode. I**n classical rotordynamics this would represent a "rigid" rotor on elastic supports**

If $1 < M^{a}_{kr}$ $/ \boldsymbol{ke^{2}} < 6.75$ then the resonance phenomenon takes place that results in destruction of rotor machine. **This state is equivalent to under-damped system with high sensitivity to unbalance and high amplitude amplification. This would represent a flexible rotor clamped in "rigid" bearings/supports.**

Therefore the minimal velocity $\boldsymbol{\omega = 1.74.\omega_{kr}}$, determines the velocity of completion of resonance phenomena. It also determines the range of operative velocities of rotor's uniform rotation up to the moment of auto-oscillations.

At the same time the minimal velocity determines the commencement of resonance phenomena at rotor's shutdown. Therefore the minimal velocity $\boldsymbol{\omega = 1.74.\omega_{kr}}$ should be considered as resonance velocity $\boldsymbol{\omega_{r}}^{s}$ at supercritical mode of rotor's rotation.

Variation of parameters relation from Figures 9, 10, 12 and 13, and variations of parameters from proportionality equation of rotor fundamental harmonic resonance in free-free state are graphically presenting rotor responses at critical velocities which are dependent on rotor mass axis eccentricity and rotor body rigidity based on its geometry and material properties.

$$\omega_{res} = \sqrt{\frac{\frac{\frac{T \triangle L}{V \rho}}{g}}{\frac{L}{2L}}} \quad (1/\text{sec})$$

where:

T = longitudinal tension (lb/in2)
$\triangle$L = elongation (non-dimensional)
V = rotor volume (in3)
ρ = Material density (lb/in3)
g = gravity acceleration (in/sec2)
W = frequency spatial wave length)
L = distance between rotor constraints

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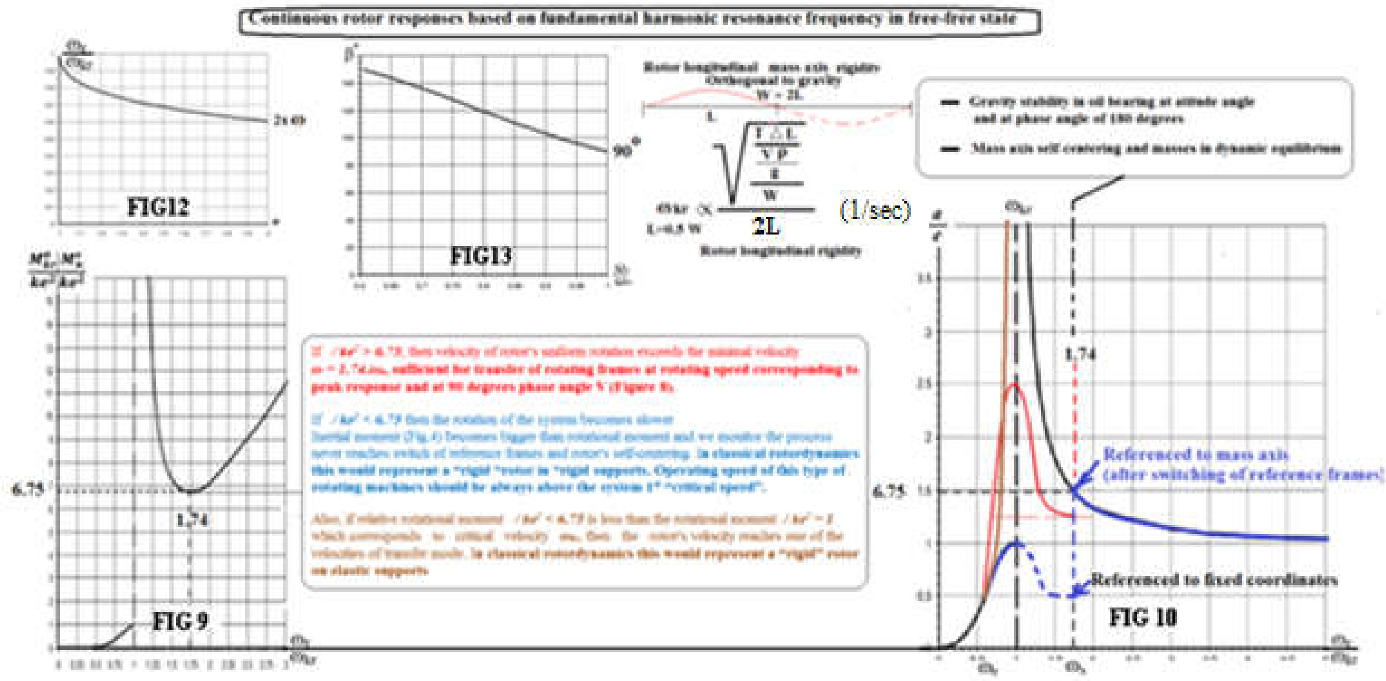

Critical velocity of the steady system rotation

Critical velocity ω_{kr} is considered in generally as the theoretical velocity since it is not possible to confirm it by experiments.
By reaching the critical velocity it creates duality of rotor's rotation parameters. At this velocity the uneven change of shaft's bend, angle of rotor's turn around shaft's middle line, inertial and rotating moment should take place. In mechanics the continuous changes of rotor's rotation parameters are impossible. Any process should be constant. Absence of resonance's constancy is explained by the fact that system of equations (1) of dynamics of Jeffcott rotor was obtained when change of velocity was very slow. The change of velocity could be neglected and we could consider that rotor rotates uniformly. In result all processes required for "rotor- shaft" system's transfer of reference frames through critical velocity is almost instantaneous. **As a result, we had a false impression that critical velocity separates non-linear response across system critical velocity frequency range which occurs at subcritical velocities, from the process of rotor's mass axis self-centering, which occurs at supercritical velocities, and vice-versa.**

Based on the above statements, it is necessary to consider the rotor dynamics and taking into account the effect of additional accelerating moment which ensures uniformly accelerated rotor's rotation. Additional accelerating moment M_{yc}^{a} can be presented as part of rotating moment M_{kr}^{a} i.e. we can consider that $M_{yc}^{a} = 0.1\, M_{kr}^{a}$ etc.

As result, the sum of moments M_{yc}^{a} and M_{kr}^{a} can be presented as $\Sigma M_{kr}^{a} = n\ M_{kr}^{a}$ where **n** >I.
The moments M_{kr} and M_{kr}^{e} will be increased in the same proportion.

System of equations of uniformly accelerated Jeffcott rotor rotation

Taking into account the above comments, let's consider the scheme of forces and moments which affect the rotor during uniformly accelerated rotation which is demonstrated in **Figure 8 and 11.**

Let's introduce to the scheme the accelerating moment $\boldsymbol{M^a_{yc}}$ which ensures uniformly accelerated rotor's rotation.

The affect of accelerating moment $\boldsymbol{M^a_{yc}}$ which acts onto rotor will result in appearance of additional inertial moment $\boldsymbol{M^a_{yc}}$. This inertial moment can be presented as force $\boldsymbol{F^u}_{yc}$ that is acting on the arm $\boldsymbol{a}$.

Let's apply the D'Alembert principle in order to transform the task of dynamics to the task of statics, i.e. let's transform the task of uniformly accelerated rotation to the task of uniform rotation of "rotor-shaft" system. At the same time at the scheme presented in Fig.5 let's demonstrate the rotational moment $\boldsymbol{M^a_{kr}}$ which ensures uniform rotation of rotor as the pair of forces $\boldsymbol{F^u_{kr}}$ which have arm $\boldsymbol{e}$. The accelerating moment $\boldsymbol{M^a_{yc}}$ is also presented as the pair of forces$\boldsymbol{F^a_{yc}}$ which have arm $\boldsymbol{e}$.

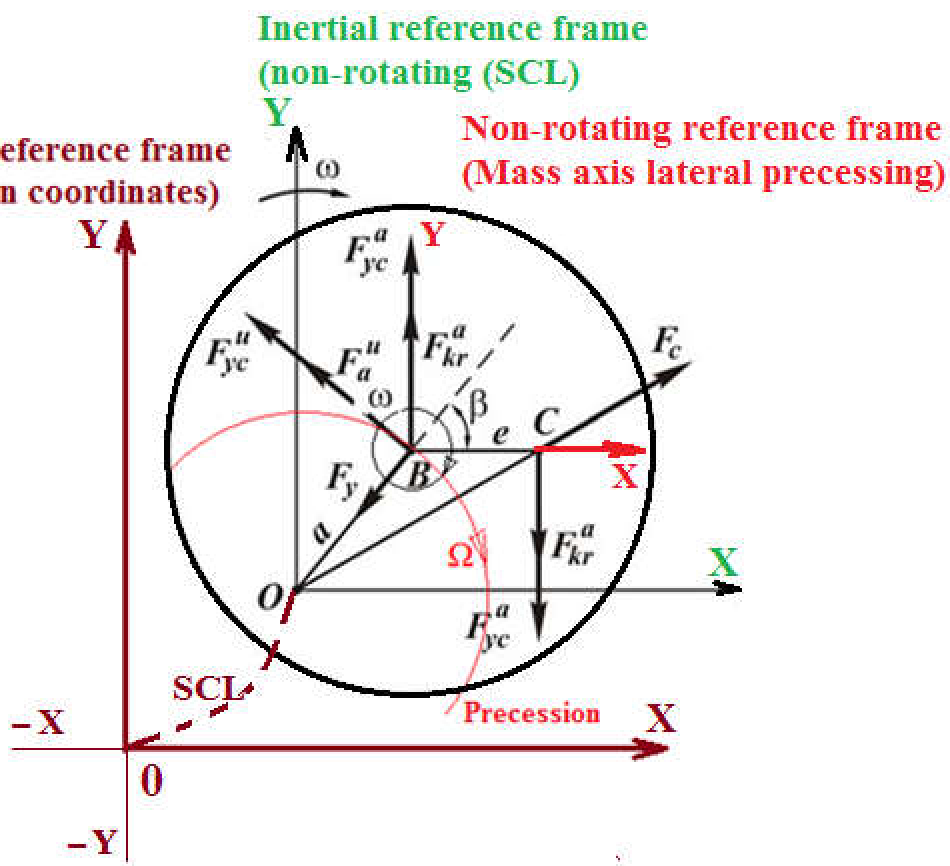

Figure11 Disc of a Continuous rotor reference frames

In order to obtain the system of equations we take into consideration the condition of rotor's dynamic balance during uniformly accelerated rotation. In the beginning let's compile the equation of moments relatively to the point $\boldsymbol{O}$ at definite instant velocity $\boldsymbol{\omega}$.

$$F_{a}^{u} + F_{yc}^{u} - F_{kr}^{a} - F_{yc}^{a} = 0 \qquad (7)$$

or

$$nM_{a}^{u} = nM_{kr}^{a} \qquad (8)$$

Since $\boldsymbol{M_{yc}^{u}} = \boldsymbol{n\, m\, \omega^2 a^2} - \boldsymbol{m\, \omega^2 a^2}$, then it follows from dependency that:

$$F_{yc}^{a} = nm\omega^2 a - m\omega^2 a = m\omega^2 a\left(n-1\right) \qquad (9)$$

Therefore there were defined all forces and moments which affect onto rotor that rotates in uniformly accelerated way around rotational axis.

Let's compile the equation of moments relative to the point $\boldsymbol{B}$ at definite instant velocity $\boldsymbol{\omega}$.

$$F_{c}\frac{ea\sin\beta}{\rho} - nM_{kr}^{a} = 0 \qquad (10)$$

or

$$m\omega^2\rho\frac{ea\sin\beta}{\rho} - nm\omega^2 a^2 = 0 \qquad (11)$$

It follows from dependency that:

$$\sin\beta = n\frac{a}{e} \qquad (12)$$

The dependency (12) shows that rotor turns around shaft's middle line at angle 90° while value of $\boldsymbol{a}$ is less than $\boldsymbol{e}$ since:

$$a = \frac{e}{n} \qquad (13)$$

Let’s compile the equation of moments relatively to the point $\boldsymbol{C}$ and definite instant velocity $\boldsymbol{\omega}$:

$$nM_{kr}^{a} + F_{a}^{u}e\cos\beta + F_{yc}^{u}e\cos\beta - F_{y}e\sin\beta = 0 \qquad (14)$$

or

$$nm\omega^2 a^2 + nm\omega^2 ae\cos\beta - kaen\frac{a}{e} = 0 \qquad (15)$$

The final view of Equation (15) will be:

$$m\omega^2\left(a+e\cos\beta\right)-ka=0 \qquad (16)$$

For transfer mode of rotor's rotation around rotational axis the equation (16) can be presented as:

$$m\omega^2\left(a+\sqrt{e^2-n^2a^2}\right)-ka=0 \qquad (17)$$

The equation (17) allows us to determine the value of shaft's bend depending on rotor's rotation velocity:

$$a=\frac{m\omega^2 e}{\sqrt{n^2+k-2km\omega^2+m^2\omega^4}} \qquad (18)$$

The equation (17) also allows us to determine the value of angular velocity when the condition of dynamic balance is ensured:

$$\omega=\sqrt{\frac{ka}{m\left(a+\sqrt{e^2-n^2a^2}\right)}} \qquad (19)$$

As it follows from the dependency, the critical velocity of rotor's rotation ω_{kr} is achieved in case if:

$$e^2-n^2a^2=0,\ \text{i.e. } a=\frac{e}{n}\ .$$

Let us refer to graph of dependency of relative shaft's bend on relative velocity shown in **Figure 9**. The graph indicates the velocity of rotor's rotation when the resonance phenomenon occurs. Let's call it as resonance velocity ω_r on transfer rotation mode. Since ***a/e = 1/n*****,** then resonance velocity ω_r is always less than critical velocity ω_{kr}.
In **Fig.12** the dependency of relative resonance velocity on value n is presented. It was compiled applying graph in **Figure 9.**

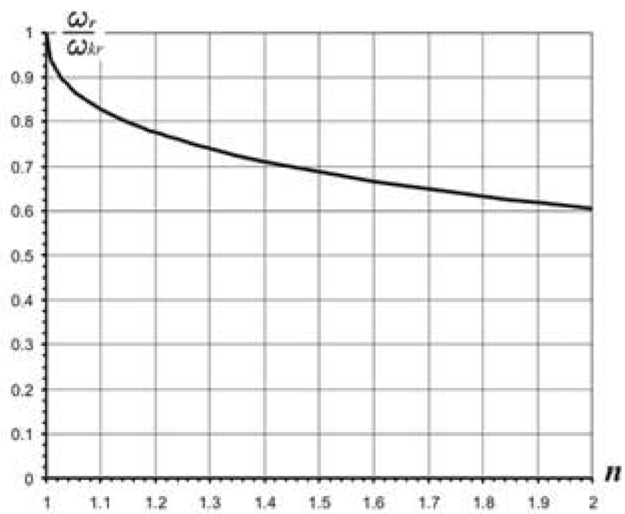

Figure 12

The **Figure 13** presents dependency of rotor's phase angle $\boldsymbol{\beta}$ around shaft's middle line relative to resonance velocity.

This dependency shows that angle $\boldsymbol{\beta}$ is less than 90° for any resonance velocity $\boldsymbol{\omega_r}$ at the moment of resonance phenomena commencement. Therefore, in order to ensure conditions for rotor's self-centering at preset resonance velocity $\boldsymbol{\omega_r}$ then rotor should be rotated beyond the angle of **180°-** $\boldsymbol{\beta}$**.**

At the same time the angle $\boldsymbol{\beta}$ should correspond to preset resonance velocity.
The conducted research shows that accelerating moment determines the resonance velocity $\boldsymbol{\omega_r}$ and angle of rotor's turn around shaft's middle line at the moment of resonance phenomena commencement.

In summary, the conducted research defines the range of velocities which is connected with resonance phenomenon. This range of velocities lies between resonance velocity $\boldsymbol{\omega_r}$ of transfer rotation mode and resonance velocity $\boldsymbol{\omega_r^s = 1.74\ \omega_{kr}}$ of supercritical rotor's rotation mode.

Finally, the conducted research explains the physics of resonance phenomenon and effect of rotor's mass axis self- centering above critical velocity range.

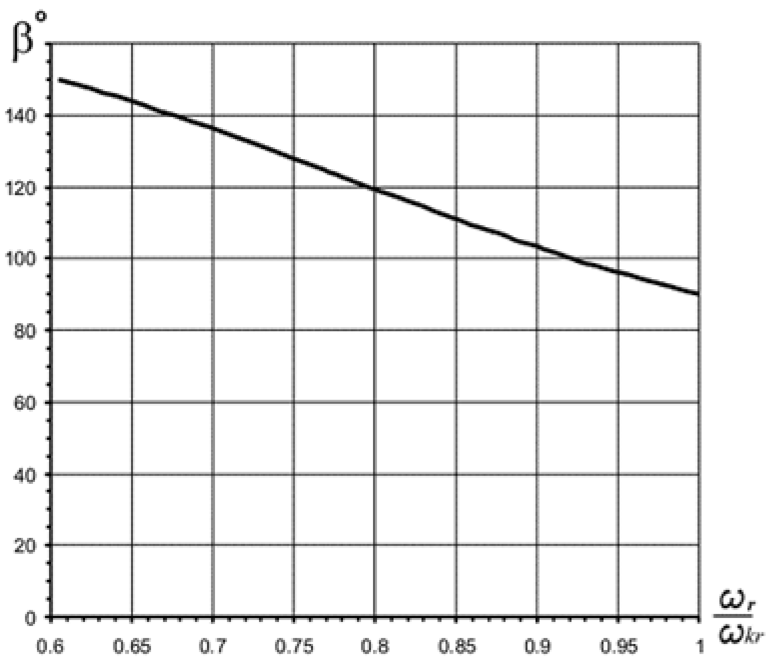

Figure 13

Physics of resonance effect

In the theory of rotor dynamics the resonance is considered as the phenomenon which is similar to resonance phenomenon in theory of oscillations. Appearance of resonance is explained by coincidence of rotational velocity of "rotor-shaft" system with critical velocity rotor critical velocity is a synonym of rotor inherent fundamental harmonic resonance frequency in free-free state, dependent on rotor material properties ratio of longitudinal rigidity over its mass. However the resonance in rotor dynamics considerably differs from resonance of linear harmonic oscillation processes. This is the resonance phenomenon that creates conditions for final rotor's turn around shaft's middle line when rotor's center of mass is located between geometrical axis and rotation axis. In result rotor and shaft which were rotating around rotational axis with their heavy side radially outwards, after resonance they rotate with their lightweight side radially outwards.

In order to explain the peculiarities of resonance in rotor dynamics examine **Figure11** with a reference to the action of moments relatively to rotor's center of mass.

The moments that were generated by the pairs of forces $\boldsymbol{F^{a}_{kr}}$ and $\boldsymbol{F^{a}_{yc}}$ strive for overcoming the shaft's elasticity force moment $\boldsymbol{F_{y}}$ that is defined relative to the rotor's center of mass. However, until the projection of forces $\boldsymbol{F^{a}_{kr}}$ and $\boldsymbol{F^{a}_{yc}}$, on the line of action of shaft's elasticity force $\boldsymbol{F_{y}}$, is equal to elasticity force, rotor will achieve the state of natural rotation around its mass axis relative to the shaft's middle line when it reaches angle of $\boldsymbol{\beta \sim 90°}$.

When rotor's velocity exceeds resonance velocity $\boldsymbol{\omega_r}$, the value of forces vectors $\boldsymbol{F^a_{kr}}$ and $\boldsymbol{F^a_{yc}}$, starts to exceed the value of shaft's elasticity force $\boldsymbol{F_y}$. The elasticity force is unable to retain rotor relatively to the rotational axis. At the same time the increase of rotor's velocity assists in “unlimited” moving-away of rotor from rotational axis. The center of mass and point of rotor's mounting to the shaft are moving radially away from rotational axis. This phenomenon takes place until the moment when rotor turns by an angle of $\boldsymbol{\beta} = \boldsymbol{90°}$ around shaft's middle line.

In order to understand the reason of rotor's turn let's consider the affect of moments relatively to the geometric center of rotor (**point** $\boldsymbol{B}$). The moments generated by the pairs of forces $\boldsymbol{F^a_{kr}}$ and $\boldsymbol{F^a_{yc}}$ are striving to turn rotor around shaft's middle line overcoming the moment that is generated by centrifugal force around **point** $\boldsymbol{B}$**.** At rotor's velocity that exceeds resonance velocity $\boldsymbol{\omega_r}$ the moment generated by centrifugal force is not able to counteract to the moments generated by forces$\boldsymbol{F^a_{kr}}$ and $\boldsymbol{F^a_{yc}}$. Therefore rotor continues to turn around shaft's middle line. At sufficient power of driving gear the alignment of center of mass with a line of shaft's elasticity force action takes place. New location of rotor's center of mass creates conditions for rotor's self- centering and self-balancing (Threshold of gravity stability and dynamic equilibrium in state of minimum action). At same time, the rotor amplitude is descending relative to rotating axis.. If the power of driving gear is not sufficient to overcome the rotor inertia, then resonance phenomena result in unstable mode of rotor's rotation and destruction of rotor and machine.

The peculiarities of rotor dynamics during transfer through critical velocity are well demonstrated by the results of experiments presented on the Fig.14.

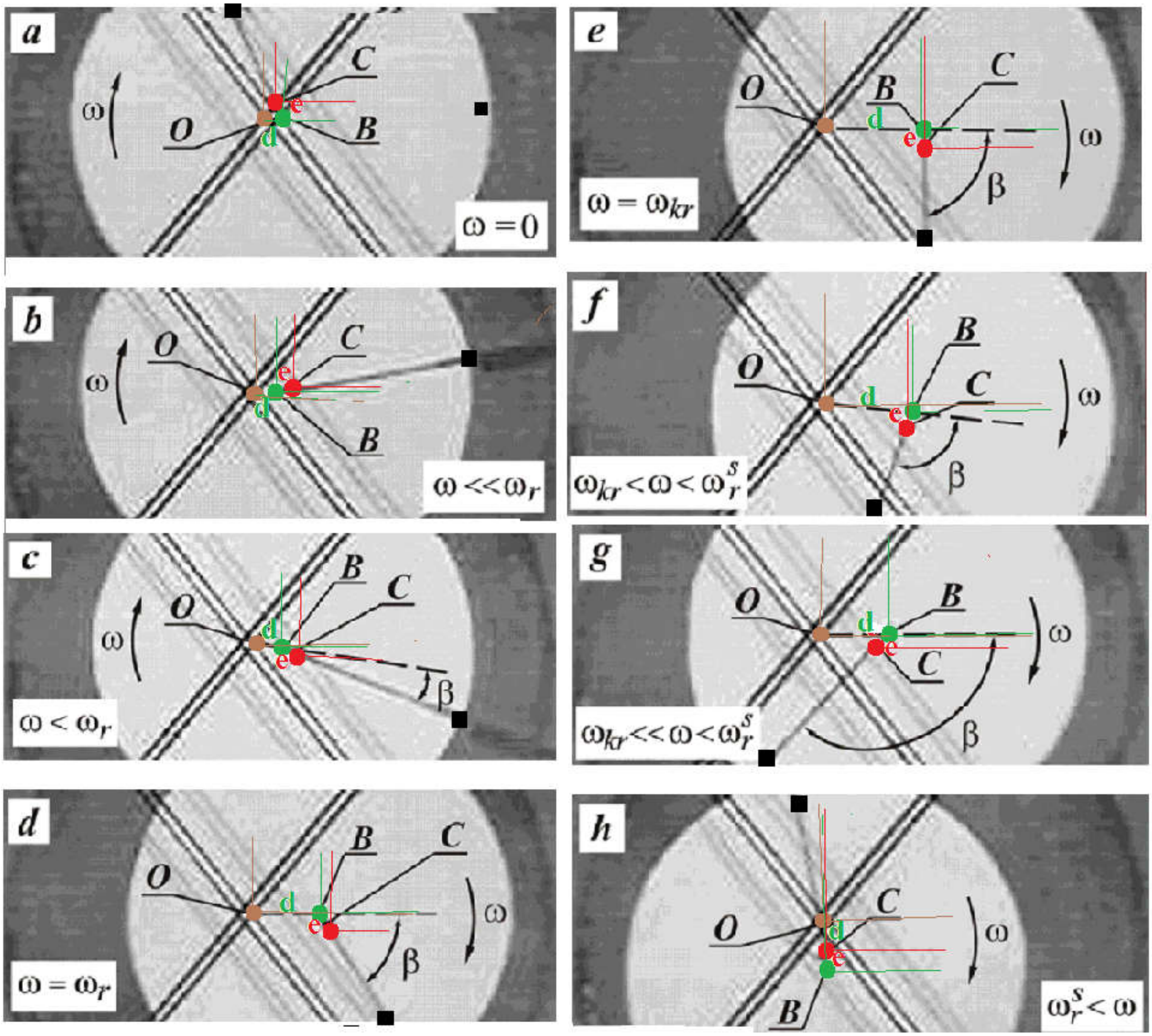
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Figure 14

Conclusion

1. Study has shown that **1st critical speed** of the system with flexible supports **forced resonance frequency response is not synonymous to** rotor linear oscillating fundamental harmonic **resonance frequency** in a free state in gravity environment.

2. The operating speed frequency, and the efficiency of an operating horizontal rotating machine with hydrodynamic bearings, consisting of "rigid" or "flexible" rotor, constrained by gravity in two bearings within a finite "L" distance between them, should not be higher than rotor natural harmonic resonance frequency, in order to avoid the situation that any external pulsating excitation force would excite fundamental harmonic resonance frequency (3^{rd} mode) and could destabilize rotor operating in oil bearings, and excite subsynchronous whirling at speed of two times system 1^{st} critical speed.

3. For a stable rotor operation at rated speed and load, in rotating machine system, the total system stiffness and mass including rotor, bearing oil and bearing support, must have 1st critical resonance response at system critical velocity 10% to 20% lower, than 0.5x of system operating speed. At the same time a resultant radial deviation from mass eccentricities, between journals axis and rotor body centroidal mass axis must be brought to "balance"(symmetry) on balancing machine within allowable limits according to ISO1940-1, for selected speed of the system 1^{st} critical velocity. Centrifugal forces generated from axially distributed eccentric mass axis, and dynamic mass axis created from radial and axial torque moments relative to gravity constraints from three correction masses proportionally distributed axially on rotor body between constraints, must create a virtual mass axis that is practically coincident with rotating journals axis.

4. Operating speed of machine with rigid rotor on a rigid supports must be lower than system 1^{st} critical speed. That way, the switching of non-rotating reference frame never takes place, and any "unbalance" can be resolved by balancing the rotor in two planes.

5. When machine with rigid rotor on rigid bearings operating speed is higher than system 1^{st} critical speed, bearings supports must be devised in way to allow mass axis self centering, (e.g. "squeeze film dampers", magnetic bearings, hydrostatic bearings or lattice springs), to avoid damaging to bearings and structure.

For an approach to balancing, when viewing turbine and generator rotors as a continuous solid body, operating in hydrodynamic oil bearings, constrained on elastic supports by gravity, it has been proven in practice that for rotors for steam and combustion turbine and generator sets over ~40 MVA, it is more economically beneficial to balance the rotor mass axis eccentricities on high speed balancing machines based on displacements at speed equivalent to just below 90 degrees phase of the system 1^{st} critical velocity, with weights placed simultaneously in three planes, distributed axially proportionally to size of response displacement vectors measured at at journals. With the added balance masses forming a dynamic mass axis of their own, when the rotor is accelerated, this dynamic mass axis will create a "virtual" mass axis against the rotor's inherent static mass axis, effectively simulating a concentric overall mass axis coincident with the journal axis (within a reasonable, machining allowable tolerance, according to modified interpretation of the ISO 1940-1). Rotor will remain and operate as balanced at any speed. The minimum of three planes is necessary for balancing, following the 2N+1 formula for the number of nodal points (anti-node and nodes) defining the purely rigid modal elements segments of the eigenvector of a flexible rotor, or a rigid bowed rotor at a fundamental null mode. Two additional planes may be required for trim balancing of residual "unbalances" at operating speed and overspeed, for very flexible rotors. Considering that rotor vibration is rooted in eccentricity of the mass axis of a continuous rotor, reducing vibration should be viewed as compensating mass axis eccentricity and restoring symmetry of rotor body mass about the rotor journal centerline axis. Mass axis eccentricity relative to journals centerline axis and corresponding responses CAN NOT BE RESOLVED BY "BALANCING" for all speeds.

6. When rotor is accelerated above supercritical velocities, or at constant operating velocity with increasing machine load, mass axis, as non-rotating reference frame is **continuously self-centering** in space, forced by inertia forces from the residual axially distributed mass eccentricities, or from asymmetric torque from external sources. Journals centers during mass axis self-centering at same time are whirling as gyroscopic rotating pendulums, pinned at COM, and flexibly connected at respective bearings and supports. Visual presentation of journals motions from sensors referenced to global coordinates is interpreted as "rocking" motion of 2^{nd} critical speed.

"Balancing" the response of such rotors, with axially asymmetric masses, at supercritical speeds would not be necessary if balancing on balancing machine was done in three planes at subcritical speed.

7. When balancing on balancing machine, turbine and generator rotors with long flexible coupling overhang, it is necessary for balancer to determine if additional bearing support is necessary during balancing process. (Ref.: "Long-Overhang rotor vibration estimation: an Approximate Approach", 2004.)

8. Balancing shaft sections between bearings on an assembled turbine generator set, there must be available three balancing planes, and balancing should performed in a same manner as balancing rotor solo on high speed balancing machine.

9. Therefore the objective of balancing rotors is to vanish the bearings dynamic reaction forces, and reduce journals radius of orbital motion at 1^{st} critical speed, and consequently at operating speed, to magnitude of internal, inherent mass axis eccentricity (defined as permissible eccentricity from ISO 1940-1, with reference to speed of system 1^{st} critical speed). Accordingly, balancing methods of large turbine and generator rotors should be balanced, not based on axially distributed point mass "unbalance" force vectors, exciting particular mode eigenvectors, but based on compensating the inherent longitudinal mass axis between bearings constraints, with rotor in state of rest, by dynamic radial forces and axial moments, across rotor COM, compensating axially distributed masses in three planes, forming a dynamic mass axis mirror imaging the internal, inherent, mass axis (line intersecting centers of masses) of rotor in state of rest (Null mode harmonic).

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